# Chapter 13

# Application: Implicit Cartels

This chapter discusses many important subgame-perfect equilibrium strategies in optimal cartel, using the linear Cournot oligopoly as the stage game. For game theory they provide many applications of single-deviation principle in repeated games. The first strategy is the simple trigger strategy, that switches to the myopic Nash equilibrium forever after any deviation. I first characterize the range of discount factors under which the monopoly prices can be supported by such a subgame-perfect equilibrium. Then, I find the optimal production supported by such a subgame-perfect equilibrium for any given discount factor. Next I study the Carrot & Stick strategies that reward the good behavior by switching to Carrot state and punish the bad behavior by switching to the Stick state. Here, in the Stick state, the firms can inflict painful punishments, which can be costly to themselves, by fearing that the failure to punish will prolong the punishment and delay the reward at the end. Finally, I consider a variation of the Carrot & Stick strategy to discuss the price wars.

## 13.1   Infinitely Repeated Cournot Oligopoly

I will use the infinitely repeated linear Cournot oligopoly as the main statel of a cartel. There are $n$ firms, each with marginal cost $c \in (0, 1)$. In the stage game, each firm $i$ simultaneously produce $q_i$ units of a good and sell it at price

$$P = \max\left\{1 - Q, 0\right\}$$

where $Q = q_1 + \cdots + q_n$ is the total supply. In the repeated game, all the past production levels of all firms are publicly observable, and each firm's utility function is the discounted sum of its stage profits, where the discount factor is $\delta$:

$$u_i = \sum_{t=0}^{\infty} \delta^t q_{i,t} \left( P\left( q_{1,t} + \cdots + q_{n,t} \right) - c \right),$$

where $q_{j,t}$ is the production level of firm $j$ at time $t$. Sometimes it will be more convenient to use the discounted average value, which is $(1 - \delta) u_i$.

For any $q$, write

$$f(q) = q\left( P(nq) - c \right) = q\left( \max\left\{ 1 - nq, 0 \right\} - c \right) \tag{13.1}$$

for the (daily) profit of a firm when each firm produces $q$ and

$$g(q) = \max_{q'} q\left( P\left( q' + (n-1)q \right) - c \right) = \begin{cases} \left( 1 - (n-1)q - c \right)^2 / 4 & \text{if } (n-1)q \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{13.2}$$

for the maximum profit of a firm from best responding when all the other firms produce $q$.

## 13.2   Monopoly Production with Patient Firms

If it is possible to enforce, it is in the firms' best interest to produce the monopoly production level

$$Q^M = 1/2$$

in total and divide the revenues according to their favored division rule, which could be attained by assigning some production levels to the firms that add up to $Q^M$. For the sake of simplicity, let us assume that they would like to divide it equally. Then, the above outcome is attained by simply each firm producing

$$q^M = Q^M / n = (1 - c) / (2n).$$

As it has been established by the Folk Theorem, when the discount factor is high, such outcomes can be an outcome of a subgame-perfect equilibrium. In that case, the firms can make some tacit informal plans that form a subgame-perfect equilibrium and yield

the desired outcome. Since the plan is a subgame-perfect equilibrium they may hope that everybody will follow through in the absence of an official enforcement mechanism, such as courts.

A simple strategy profile that leads to the above outcome is as follows:

> **Simple Trigger Strategy:** Each firm is to produce $q^M$ until somebody deviates, and produce $q^{NE} = (1 - c) / (n + 1)$ thereafter.

The above strategy profile yields each firm producing $q^M$ forever, stipulating that they would fall back to the myopic Nash equilibrium production $q^{NE}$ if any firm deviates, leading to the breakdown of the cartel. This strategy profile may or may not be a subgame-perfect equilibrium, depending on the discount factor. This section is devoted to determine the range of discount factors under which it is indeed a subgame-perfect equilibrium.

Once a firm deviates and the cartel breaks down, the firms are playing the stage-game Nash equilibrium regardless of what happens thereafter, which is a subgame-perfect Nash equilibrium of the subgame after break down, as it has been established before. Hence, by the single-deviation principle, it suffices to check whether a firm has an incentive to deviate while the cartel in place (i.e., no firm has deviated from producing $q^M$). In that case, according to the single deviation test, the average discounted value of producing $q^M$ for a firm $i$ is

$$V_C = f\left(q^M\right) = \frac{(1 - c)^2}{4n}.$$

A deviation of producing $q \neq q^M$ yields the average value of

$$V_D\left(q\right) = (1 - \delta) q \left(1 - \frac{n - 1}{2n} - q - c\right) + \delta f\left(q^{NE}\right),$$

where the first term is the payoff from the current period, in which the other firms are producing $q^M$ each, and the second term $f\left(q^{NE}\right) = (1 - c)^2 / (n + 1)^2$ is the value of flow payoff of Nash equilibrium, starting from the next day. The best possible deviation payoff is

$$V_D^* = \max_{q \neq q^M} V_D\left(q\right) = (1 - \delta) g\left(q^M\right) + \delta f\left(q^{NE}\right),$$

where $g\left(q^M\right) = \left(\frac{n+1-2nc}{4n}\right)^2$ is the profit from best responding to $q^M$. The firm does not have an incentive to deviate if and only if
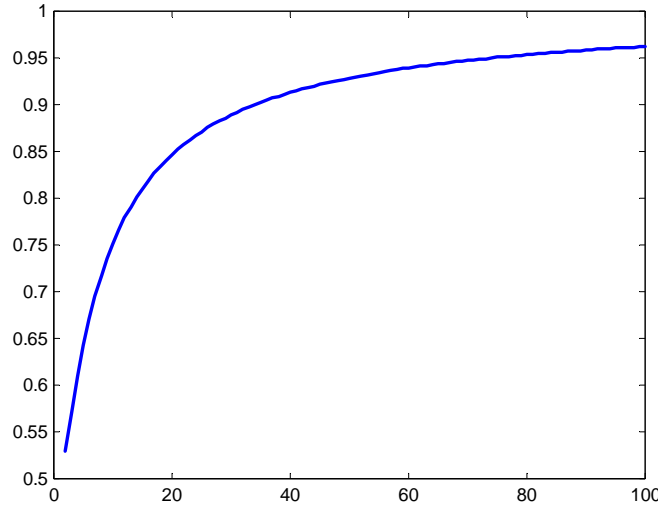
$$V_C \geq V_D^*,$$

Figure 13.1: $\delta^{STS}$ as a function of $n$.

i.e.,

$$\delta \geq \delta^{STS} \equiv \frac{g\left(q^M\right) - f\left(q^M\right)}{g\left(q^M\right) - f\left(q^{NE}\right)}.$$

Clearly, for any $n$, $\delta^{STS}$ is less then 1, and hence the simple trigger strategy profile above is a subgame perfect equilibrium when the discount factor is large (larger than $\delta^{STS}$). As shown in Figure 13.1, for small $n$, $\delta^{STS}$ is reasonably small, and the monopoly prices are maintained in the simple trigger strategy equilibrium for reasonable values of $\delta$. On the other hand, $\delta^{STS}$ is increasing in $n$, and $\delta^{STS} \to 1$ as $n \to \infty$. Hence, for any given discount factor, as the number of firms becomes very large, the simple trigger strategy profile fails to be an equilibrium.

## 13.3   Optimal Production Level with a Fixed $\delta$

For a fixed $\delta$ and $n$ with $\delta < \delta^{STS}$, the simple trigger strategy above is not an equilibrium when the firms tries to maintain the monopoly prices on the path. Such a plan may likely to tempt the firms to over produce in equilibrium, breaking the cartel, and resulting in highly competitive outcome with low prices and profits. The firms may want to target a lower profit that can be supported by a simple trigger strategy equilibrium. This section

is devoted to find the optimal production level supported by a simple trigger strategy.

More precisely, for a fixes $\delta$ and $n$, consider the following strategy profile:

**Simple Trigger Strategy $(q^*)$:** Each firm is to produce $q^*$ until somebody deviates, and produce $q^{NE} = (1 - c) / (n + 1)$ thereafter.

Note that in the outcome of this strategy profile each firm produces $q^*$ at each day, yielding the average discounted value of

$$V_C (q^*) = f (q^*) = q^* (1 - nq^* - c) \tag{13.3}$$

to each firm. The main question is: Which $q^*$ maximizes the firms' profits $V_C$ subject to the constraint that the simple trigger strategy profile is a subgame-perfect Nash equilibrium?

Once again, since the myopic Nash equilibrium is played after the breakdown of the cartel, it suffices to check that there is no incentive to deviate on the path, in which all firms produced $q^*$ at all times. At any such history, any unilateral deviation $q \neq q^*$ yields the average discounted value of

$$V_D (q) = (1 - \delta) q (1 - (n - 1) q^* - q - c) + \delta f \left(q^{NE}\right)$$

to the deviating firm. To see this, note that in the first day, the firm's profit is $q (1 - (n - 1) q^* - q - c)$ as it produces $q$ and all the other firms produce $q^*$. This one time profit is multiplied by $(1 - \delta)$. After the deviation, the firm gets the myopic Nash equilibrium profit of $f \left(q^{NE}\right) = (1 - c)^2 / (n + 1)^2$ every day, which has the average discounted value of $f \left(q^{NE}\right)$. Since the firm gets this starting the next day, it is multiplied by $\delta$. The simple trigger strategy profile above is a subgame perfect Nash equilibrium if and only if

$$V_C (q^*) \geq V_D (q) \qquad (\forall q \neq q^*).$$

This constraint reduces to

$$V_C (q^*) \geq \max_{q \neq q^*} V_D (q) = (1 - \delta) g (q^*) + \delta f \left(q^{NE}\right); \tag{13.4}$$

the simple trigger strategy profile is a subgame-perfect equilibrium if and only if (13.4) is satisfied. Hence, the objective in this section is to maximize $V_C (q^*) = f (q^*)$ in (13.3) subject to the constraint $f (q^*) \geq (1 - \delta) g (q^*) + \delta f \left(q^{NE}\right)$ in (13.4).

When $\delta > \delta^{STS}$, the monopoly production $q^M$ is an equilibrium value for $q^*$. (After all, it has been shown in the previous section that the simple trigger strategy for $q^* = q^M$ is a subgame-perfect equilibrium if and only if $\delta \geq \delta^{STS}$.) In that case, the optimal value for $q^*$ is $q^M$. When $\delta < \delta^{STS}$, $tq^M$ is not an equilibrium value for $q^*$. In that case, the minimum allowable value for $q^*$ is optimal, which is given by the equality

$$f\left(q^*\right) = \left(1 - \delta\right) g\left(q^*\right) + \delta f\left(q^{NE}\right),$$

i.e.,

$$q^* \left(1 - nq^* - c\right) = \left(1 - \delta\right) \frac{\left(1 - (n-1) q^* - c\right)^2}{4} + \delta \left(1 - c\right)^2 / \left(n + 1\right)^2.$$

The explicit solution to the above quadratic equation is not important. The effect of the parameters on the solution can be gleaned from the equation. The left-hand side is independent of the discount factor, while the expression on the other side is decreasing in $\delta$. This is because the payoff from deviation, which is multiplied by $(1 - \delta)$, is larger than the myopic Nash equilibrium payoff, which is multiplied by $\delta$. Hence, as the discount factor increases the right hand-side goes down, decreasing $q^*$. This results in lower amount of production and higher amounts of profits, in the expense of the consumers. This is because more patient firms can maintain higher cartel prices without being tempted by the short-term opportunities.

## 13.4 Reward and Punishment: Carrot-Stick Strategies

In the above strategy profiles, the level of equilibrium quantities are limited by the fact that the punishment after a deviation resorts to Nash equilibrium of the stage game, which limits the deviators' payoffs from below. In many games like the Cournot oligopoly, the average payoff of a player in the repeated game can be lower than his lowest equilibrium payoff in the stage game. Using such low SPE payoffs after a deviation, one can maintain even higher equilibrium payoffs in a SPE. Such equilibria are of course more sophisticated than the simple trigger strategies employed in the previous section. Among such equilibria a relatively simple Carrot&Stick strategy plays a central role. This section is devoted to constructing such a Carrot and Stick strategy in Cournot oligopoly.

**Carrot & Stick Strategy:** There are two states: Carrot and Stick. Each player plays $q_C$ in Carrot state and $q_S$ in Stick state. The game starts in Carrot state. At any $t$, if all players play what they are supposed to play, they go to Carrot state at $t + 1$; they go to Stick state at $t + 1$ otherwise.

In a Carrot&Stick strategy, the Carrot state is used as a reward for following through and the Stick state is used as a punishment for deviation. Hence, the profit from $(q_S, \ldots, q_S)$ is lower than the profit from $(q_C, \ldots, q_C)$. Note that punishment in the Stick state can be costly for everyone including the other players who are punishing the deviant player. They may than forgive the deviant in order to avoid the cost. In order to deter them from failing to punish the deviant, equilibrium prescribes that they, too, will be punished the next period if they fail to punish today.

The average discounted payoff from the Carrot state is

$$V_C = f(q_C), \tag{13.5}$$

and the average discounted payoff from the Stick state is

$$V_S = (1 - \delta) f(q_S) + \delta V_C = (1 - \delta) f(q_S) + \delta f(q_C). \tag{13.6}$$

Single-deviation principle yields two constraints under which the Carrot & Stick strategy profile above is a subgame-perfect equilibrium. First, no player has an incentive to unilateral deviation in the Carrot state:

$$V_C \geq \max_{q \neq q_C} (1 - \delta) qP(q + (n - 1) q_C - c) + \delta V_S = (1 - \delta) g(q_C) + \delta V_S. \tag{13.7}$$

Here the first term $g(q_C)$ is the profit from the most-profitable deviation, which is multiplied by $1 - \delta$ as it is a single profit, and the second term $V_S$ is the average discounted payoff from switching to the Stick state next day, which is multiplied by $\delta$ because it starts the next day. By substituting the value of $V_S$ in (13.6) to (13.7), one can simplify (13.7) as

$$V_C = f(q_C) \geq \frac{1}{1 + \delta} g(q_C) + \frac{\delta}{1 + \delta} f(q_S). \tag{13.8}$$

This condition finds a lower bound on the average discounted payoff $V_C$ from Carrot: it has to be at least as high as the daily profit from deviation, multiplied by $1/(1 + \delta)$, and the daily profit at the Stick state, multiplied by $\delta/(1 + \delta)$.

The second constraint is that no firm has an incentive to deviate unilaterally in the Stick state:

$$V_S \geq \max_{q \neq q_S} (1 - \delta) \, qP \left(q + (n-1) \, q_S - c\right) + \delta V_S = (1 - \delta) \, g \left(q_S\right) + \delta V_S. \qquad (13.9)$$

That is, applying the possibly painful punishment at the Stick state must be at least as good as deviating from this for one day and postponing it to the next period. This constraint simplifies to

$$V_S \geq g \left(q_S\right). \qquad (13.10)$$

That is, the average discounted payoff in the stick state is at least as high as the daily profit from deviation at that state. By substituting the value of $V_S$ from (13.6), one can write this directly, again, as a lower bound on the equilibrium profit:

$$f \left(q_C\right) \geq g \left(q_S\right) / \delta - (1 - \delta) \, f \left(q_S\right) / \delta. \qquad (13.11)$$

The Carrot & Stick gives a subgame-perfect equilibrium if and only if the simple constraints (13.8) and (13.11) are satisfied.

In general one can obtain high values for selecting the punishment profit $f \left(q_S\right)$ very low even negative. When the costs are zero (i.e., $c = 0$), since the price is non-negative, the lowest payoff is also zero, and it is obtained from selecting $q_S = 1/ (n-1)$. In that case, $f \left(q_S\right) = g \left(q_S\right) = 0$, and the constraint (13.11) is satisfied for all $q_C$. Hence, this value of equilibrium leads to a subgame perfect equilibrium if and only if (13.8) is satisfied:

$$f \left(q_C\right) \geq \frac{1}{1 + \delta} g \left(q_C\right).$$

When this inequality is satisfied at $q_C = q^M$, then an optimal Carrot & Stick strategy for the firms is $q_C = q^M = 1/ (2n)$ and $q_S = 1/ (n-1)$. This is the case when $\delta \geq g \left(q^M\right) / f \left(q^M\right) - 1$. Otherwise, an optimal Carrot & Stick strategy is given by $q_S = 1/ (n-1)$ and $q_C$ as the smallest solution to the quadratic equation $(1 + \delta) f \left(q_C\right) = g \left(q_C\right)$.

When the marginal cost is positive (i.e., $c > 0$), one can make $f \left(q_C\right)$ negative and as small as needed by selecting a large $q_S$. In that case, the firms can inflict arbitrarily painful punishments on the deviating firm. They do so by fearing that failure of punishment only delay the punishment and the subsequent reward one more period. Giving incentive to such punishment puts an upper bound on $q_S$ through (13.11). This

upper bound is large when the marginal cost $c$ is small. I will next describe the optimal strategy for small values of $c$ so that one can choose $q_S > 1/(n-1)$. In that case, in the the Stick state, the profit is $f(q_S) = -cq_S$, i.e., the firms simply incur the cost of the production as a loss, and the optimal deviation is to avoid this loss by producing nothing, i.e., $g(q_S) = 0$. Hence, the optimal Carrot & Stick strategy maximizes $f(q_C)$ subject to the constraints

$$f(q_C) \geq \frac{1}{1+\delta}g(q_C) - \frac{\delta}{1+\delta}cq_S, \tag{13.12}$$
$$f(q_C) \geq (1-\delta)cq_S/\delta. \tag{13.13}$$

A careful reader can check that one can select the second weak inequality as equality. (That inequality can be strict only when both inequalities are satisfied at the global optimum $q_C$.) That is, one can select $cq_S = \delta f(q_C)/(1-\delta)$. In that case, the first inequality reduces to

$$f(q_C) \geq (1-\delta)g(q_C).$$

Therefore, when $\delta \geq 1 - f(q^M)/g(q^M)$, an optimal Carrot & Stick strategy is given by $q_C = q^M$ and $cq_S = \delta f(q^M)/(1-\delta)$. The firms produce the monopoly outcome, and any deviation leads to the production of $q_S$ that offsets the gain from optimal deviation. When $\delta < 1 - f(q^M)/g(q^M)$, the constraint in the last displayed inequality is binding, and the production $q_C$ in the optimal Carrot & Stick strategy is the smallest solution to the quadratic equation

$$f(q_C) = (1-\delta)g(q_C).$$

In a Carrot & Stick equilibrium, the firm produce large amounts yielding very small prices in order to punish deviations from the equilibrium. For example, in the optimal strategy above, the price becomes zero after a deviation. This can viewed as a price war.

## 13.5 Price Wars

The price wars in Carrot & Stick strategies above are supposed to last only one period. In general, the price wars can take much longer in other forms of equilibria, in which there are multiple Carrot states. This section is devoted to analysis of such subgame-perfect equilibria.

**Price War:** There are $K+1$ states: Cartel, $W_1, \ldots, W_K$. Each firm produces $q_C$ in Cartel state and $q_W = 1/(n-1)$ in states $W_1, \ldots, W_K$. The game starts at Cartel state. If each firm produces the above amounts ($q_C$ in Cartel state and $1/(n-1)$ in other states), then Cartel and $W_K$ transition to Cartel and $W_k$ transitions to $W_{k+1}$ for all $k < K$. They go to $W_1$ in the next period otherwise.

On the path of the above strategy profile, the firms produce the cartel production $q_C$ everyday. Any deviation from this production level starts a price war that lasts $K$ days. During the price war, the price is 0. If a firm is to deviate at any date during the punishment, the punishment starts all over again in order to punish the newly deviating firm.

Note that the average discounted profit at the cartel state is

$$V_C = f(q_C),$$

and the average discounted profit at $W_k$ state is

$$V_k = -\left(1 - \delta^{K-k+1}\right) cq_W + \delta^{K-k+1} f(q_C), \tag{13.14}$$

where $c$ is the marginal cost. Note that, assuming $f(q_C) \geq 0$, the situation improves as they leave more war dates in the past and get closer to the start date of the cartel with positive payoffs:

$$V_K \geq V_{K-1} \geq \cdots \geq V_1.$$

In order to check that this is a subgame-perfect equilibrium, one needs to apply the single deviation test at each state, leading to $K + 1$ constraints. First, the single-deviation test at the cartel state requires that the firms do not have incentive to deviate in the cartel state and start a price war:

$$f(q_C) \geq (1 - \delta) g(q_C) + \delta V_1, \tag{13.15}$$

i.e., the value of cartel is higher than one period optimal deviation and the value of starting a war next day. As in the previous section, by substituting the value of $V_1$ from (13.14), one simplifies this constraint to

$$f(q_C) \geq \frac{1 - \delta}{1 - \delta^{K+1}} g(q_C) - \frac{\delta\left(1 - \delta^K\right)}{1 - \delta^{K+1}} cq_W. \tag{13.16}$$

In any war state $W_k$, the single-deviation test requires that a firm does not have an incentive to deviate and start the war all over again:

$$V_k \geq \delta V_1.$$

That is, the value of being in the $k$th day of war is at least as good as not producing at all and avoiding the cost of production of a good that sells at price zero for one day and starting the war all over again in the next period. Since $V_k \geq V_1$ for each $k$, this constraint is satisfied at each war period $W_k$ if it is satisfied at the first day of the war, i.e.,

$$V_1 \geq \delta V_1.$$

Therefore, the single-deviation test in the war states yields a single constraint:

$$V_1 \geq 0,$$

i.e.,

$$f(q_C) \geq \left(1 - \delta^{K-k+1}\right) c q_W / \delta^{K-k+1}. \tag{13.17}$$

In summary, the price war strategies above form a subgame-perfect equilibrium if and only if the constraints (13.16) and (13.17) are satisfied.

What is the optimal price war strategy profile for the firms? To answer this question, note that in the optimal equilibrium, one selects $V_1 = 0$ (i.e., (13.17) is satisfied with equality) in order to provide the maximal deterrence in the cartel state:

$$c q_W = \delta^{K-k+1} f(q_C) / \left(1 - \delta^{K-k+1}\right).$$

In that case, from the equivalent form (13.15), one can see that the constraint (13.16) reduces to:

$$f(q_C) \geq (1 - \delta) g(q_C).$$

This is the same constraint as the optimal Carrot & Stick equilibrium. As in there, in the optimal price war equilibrium, one selects $q_C = q^M$ when $\delta \geq 1 - f\left(q^M\right) / g\left(q^M\right)$ and $q_C$ equal to the smallest solution to the quadratic equation

$$f(q_C) = (1 - \delta) g(q_C)$$

otherwise.

## 13.6   Exercises with Solutions

1. [2010 Midterm 2] Consider the linear Cournot oligopoly above with $c = 0$. For each of the following strategy profiles, find the parameter values under which the strategy profile is a subgame-perfect equilibrium.

   (a) Each firm is to produce $q^*$ until somebody deviates, and produce $q^{NE} = 1/(n+1)$ thereafter.

   **Solution:** Just take $c = 0$ in Section 13.3. The condition is

   $$f(q^*) \geq (1 - \delta) g(q^*) + \delta f(q^{NE}),$$

   where $f(q^*) = q^*(1 - nq^*)$, $g(q^*) = (1 - (n-1)q^* - c)^2/4$, and $f(q^{NE}) = 1/(n+1)^2$.

   (b) There are two states: Cartel and War. The game starts in the Cartel state. In the Cartel state, each firm produces $q^*$. In the Cartel state, if each firm produces $q^*$, they remain in the Cartel state in the next period, too; otherwise they switch to the War state in the next period. In the War state, each firm produces $1/n$. In the War state, if each firm produces $1/n$, they switch to Cartel state in the next period; otherwise they remain in the War state in the next period, too.

   **Solution:** This is a price war strategy with one war period, or equivalently a Carrot & Stick strategy with $q_C = q^*$ and $q_S = 1/n$. The necessary and sufficient conditions for this to be a SPE are (13.8) and (13.11). Since $f(q_S) = 0$ and $g(q_S) = 1/n^2$, these conditions simplify to

   $$(1 + \delta) q^* (1 - nq^*) \geq (1 - (n-1)q^*)^2/4,$$
   $$\delta q^* (1 - nq^*) \geq \frac{1}{4n^2}.$$

2. [Midterm 2, 2007] Consider the infinitely repeated game with the following stage game (Linear Bertrand duopoly). Simultaneously, Firms 1 and 2 choose prices $p_1 \in [0, 1]$ and $p_2 \in [0, 1]$, respectively. Firm $i$ sells

   $$q_i(p_1, p_2) = \begin{cases} 1 - p_i & \text{if } p_i < p_j \\ (1 - p_i)/2 & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

units at price $p_i$, obtaining the stage payoff of $p_i q_i (p_1, p_2)$. For each strategy profile below, find the range of parameters under which the strategy profile is a subgame-perfect equilibrium.

(a) They both charge $p_i = 1/2$ until somebody deviates; they both charge $0$ thereafter.

**Solution:** After the switch, they produce $0$ forever and the future moves do not depend on the current actions. Hence, the reduced game is identical to the original stage game. Since $(0, 0)$ is a SPE of the stage game, it passes the single-deviation test at such a history. Before the switch, we need to check that

$$V_C = 1/8 \geq (1 - \delta) \cdot 1/4 + \delta \cdot 0,$$

i.e., $\delta \geq 1/2$. (Note that by undercutting a firm can get $1/4 - \varepsilon$ for any $\varepsilon > 0$.)

(b) There are $K + 1$ states: Cartel, $W_1, \ldots, W_K$. Each firm charges $p_i = 1/2$ in Cartel state and $p_i = p^*$ in War states $W_1, \ldots, W_K$ where $p^* < 1/2$. The game starts at Cartel state. If each firm charges the above prices ($1/2$ in Cartel state and $p^*$ in War states), then Cartel and $W_K$ transition to Cartel and $W_k$ transitions to $W_{k+1}$ for all $k < K$. They go to $W_1$ in the next period otherwise.

**Solution:** As in the price war with Cournot oligopoly there are two binding conditions for SPE. In the cartel state no firm should have an incentive to undercut:

$$1/8 \geq (1 - \delta)/4 + \delta \left(1 - \delta^K\right) p^* (1 - p^*)/2 + \delta^{K+1}/8,$$

i.e.,

$$\left(1 - \delta^{K+1}\right)/8 \geq (1 - \delta)/4 + \delta \left(1 - \delta^K\right) p^* (1 - p^*)/2. \tag{13.18}$$

Second, in the first day of War there is no incentive to deviate:

$$V_1 \geq (1 - \delta) p^* (1 - p^*) + \delta V_1,$$

i.e.,

$$V_1 \geq p^* (1 - p^*).$$

Here,

$$V_k = \left(1 - \delta^{K-k+1}\right) p^* \left(1 - p^*\right) / 2 + \delta^{K-k+1}/8$$

is the average discounted payoff at $W_k$. The condition deters against the deviations in which the a firm charges slightly less and gets all of the demand for a day. By substituting the value of $V_1$ in the last equality, one can simplify this condition as

$$1/4 \geq \left(1 - \delta^{-K}\right) p^* \left(1 - p^*\right). \tag{13.19}$$

Since $V_k \geq V_1$, this condition further implies that there is no incentive to deviate at other war states:

$$V_k \geq (1 - \delta) p^* \left(1 - p^*\right) / 2 + \delta V.$$

Therefore, the conditions are (13.18) and (13.19).

## 13.7   Exercises

1. [Homework 4, 2011] Consider the infinitely repeated game with linear Cournot oligopoly as the stage game and the discount factor $\delta$. In the stage game, there are $n > 2$ firms with zero cost and the inverse-demand function $P = \max\{1 - Q, 0\}$. For each strategy profile below, find the range of $\delta$ under which the strategy profile is a subgame-perfect Nash equilibrium.

   (a) At each $t$, each firm produces $1/(2n)$ until some firm produces another amount; each firm produces $1/n$ thereafter.

   (b) At each $t$, firms $1, \ldots, n$ produce $1/2, 1/4, \ldots, 1/2^n$, respectively, until some firm deviates (by not producing the amount that it is supposed to produce); they all produce $1/(n+1)$ thereafter.

   (c) There are $K + 1$ states: Cartel, $W_1, \ldots, W_K$. Each firm produces $1/(2n)$ in the Cartel state and $1/n$ in states $W_1, \ldots, W_K$. The game starts at the Cartel state. If each firm produces what it is supposed to produce in any given state, then Cartel leads to Cartel in the next period, $W_k$ leads to $W_{k+1}$ in the next period for each $k < K$ and $W_K$ leads to Cartel. In any state, if

any player deviates from what it is supposed to produce, they go to $W_1$ in the next period.

2. [Midterm 2 Make Up, 2007] Consider the infinitely repeated game with discount rate $\delta$ and the following stage game. Simultaneously, Seller chooses quality $q \in [0, \infty)$ of the product and the Customer decides whether to buy at a fixed price $p$. The payoff vector is $(p - q^2/2, vq - p)$ if customer buys, and $(-q^2/2, 0)$ otherwise, where the first entry is the payoff of the seller and $v > 0$ is a constant.

   (a) Find the highest price $p$ for which there is a SPE such that customer buys on the path everyday.

   (b) Find the set of parameters $\hat{q}$, $p$, $n$ and $\delta$ for which the following is a SPE. We have a Trade state and $n$ Waste states $(W_1, W_2, \ldots, W_n)$. In the trade state seller chooses quality $q = v$, and the buyer buys. In any Waste state, the seller chooses quality level $\hat{q}$ and the buyer does not buy. If everybody does what he is supposed to do, in the next period Trade leads to Trade, $W_1$ leads to $W_2$, $W_2$ leads to $W_3$, $\ldots$, $W_{n-1}$ leads to $W_n$, and $W_n$ leads to Trade. Any deviation takes us to $W_1$. The game starts at Trade state.

3. [Midterm 2, 2007] Consider the infinitely repeated game with the following stage game (Linear Bertrand duopoly). Simultaneously, Firms 1 and 2 choose prices $p_1 \in [0, 1]$ and $p_2 \in [0, 1]$, respectively. Firm $i$ sells

$$q_i (p_1, p_2) = \begin{cases} 1 - p_i & \text{if } p_i < p_j \\ (1 - p_i)/2 & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

units at price $p_i$, obtaining the stage payoff of $p_i q_i (p_1, p_2)$. (All the previous prices are observed, and each player maximizes the discounted sum of his stage payoffs with discount factor $\delta \in (0, 1)$.) For each strategy profile below, find the range of parameters under which the strategy profile is a subgame-perfect equilibrium.

   (a) They both charge $p_i = 1/2$ until somebody deviates; they both charge $0$ thereafter. (You need to find the range of $\delta$.)

(b) There are $n+1$ states: Collusion, the first day of war ($W_1$), the second day of war ($W_2$), ..., and the $n$th day of war ($W_n$). The game starts in the Collusion state. They both charge $p_i = 1/2$ in the Collusion state and $p_i = p^*$ in the war states ($W_1, \ldots, W_n$), where $p^* < 1/2$. If both players charge what they are supposed to charge, then the Collusion state leads to the Collusion state, $W_1$ leads to $W_2$, $W_2$ leads to $W_3$, ..., $W_{n-1}$ leads to $W_n$, and $W_n$ leads to the Collusion state. If any firm deviates from what it is supposed to charge at any state, then they go to $W_1$. (Every deviation takes us to the first day of a new war.) (You need to find inequalities with $\delta$, $p^*$, and $n$.)

4. [Selected from Midterms 2 and make up exams in years 2002 and 2004] Below, there are pairs of stage games and strategy profiles. For each pair, check whether the strategy profile is a subgame-perfect equilibrium of the game in which the stage game is repeated infinitely many times. Each agent tries to maximize the discounted sum of his expected payoffs in the stage game, and the discount rate is $\delta = 0.99$. (Clearly explain your reasoning in each case.)

(a) **Stage Game:** Linear Cournot Duopoly: There are two firms. Simultaneously each firm $i$ supplies $q_i \geq 0$ units of a good, which is sold at price $P = \max\{1 - (q_1 + q_2), 0\}$. The cost is equal to zero.

**Strategy profile:** There are two states: Cartel and Competition. The game starts at Cartel state. In Cartel state, each supplies $q_i = 1/4$. In Cartel state, if each supplies $q_i = 1/4$, they remain in Cartel state in the next period; otherwise they switch to Competition state in the next period. In Competition state, each supplies $q_i = 1/2$. In Competition state, they automatically switch to Cartel state in the next period.

(b) **Stage Game:** Linear Cournot Duopoly of part (b).

**Strategy profile:** There are two states: Cartel and Competition. The game starts at Cartel state. In Cartel state, each supplies $q_i = 1/4$. In Cartel state, if each supplies $q_i = 1/4$, they remain in Cartel state in the next period; otherwise they switch to Competition state in the next period. In Competition state, each supplies $q_i = 1/2$. In Competition state, they switch

to Cartel state in the next period if and only if both supply $q_i = 1/2$; otherwise they remain in Competition state in the next period, too.

14.12 Economic Applications of Game Theory
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