

# Chapter 9

## Backward Induction

We now start analyzing the dynamic games with complete information. These notes focus on the perfect-information games, where each information set is singleton, and apply the notion of backward induction to these games. We will assume that the game has "finite horizon", i.e., there can only be finitely many moves in any history of moves.

### 9.1 Definition

The concept of backward induction corresponds to the assumption that it is common knowledge that each player will act rationally at each future node where he moves – even if his rationality would imply that such a node will not be reached.<sup>1</sup> (The assumption that the player moves rationally at each information set he moves is called *sequential rationality*.)

Mechanically, backward induction corresponds to the following procedure, depicted in Figure 9.1. Consider any node that comes just before terminal nodes, that is, after each move stemming from this node, the game ends. (Such nodes are called pen-terminal.) If the player who moves at this node acts rationally, he chooses the best move for himself at that node. Hence, select one of the moves that give this player the highest payoff. Assigning the payoff vector associated with this move to the node at hand, delete all the

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<sup>1</sup>More precisely: at each node  $i$  the player is certain that all the players will act rationally at all nodes  $j$  that follow node  $i$ ; and at each node  $i$  the player is certain that at each node  $j$  that follows node  $i$  the player who moves at  $j$  will be certain that all the players will act rationally at all nodes  $k$  that follow node  $j$ ,...ad infinitum.

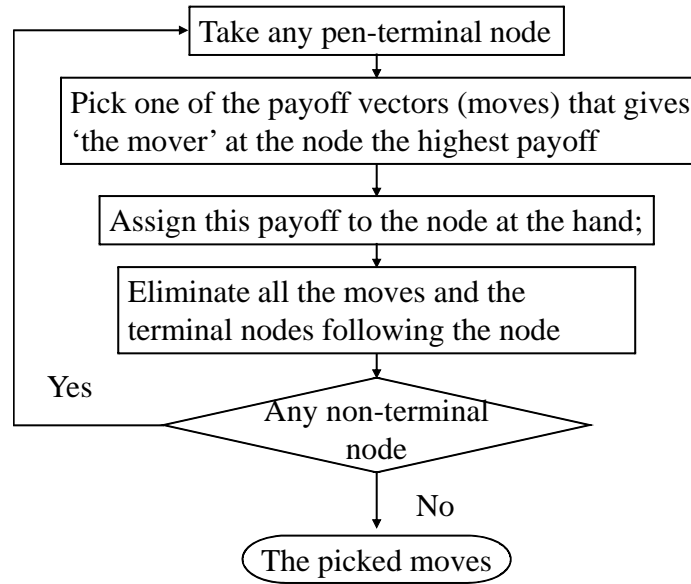
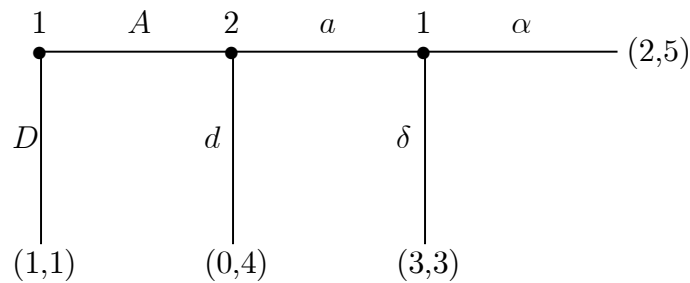


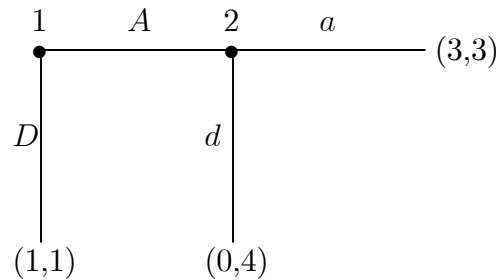
Figure 9.1: Algorithm for backward induction

moves stemming from this node so that we have a shorter game, where the above node is a terminal node. Repeat this procedure until the origin is the only remaining node. The outcome is the moves that are picked in the way. Since a move is picked at each information set, the result is a strategy profile.

For an illustration of the procedure, consider the game in the following figure. This game describes a situation where it is mutually beneficial for all players to stay in a relationship, while a player would like to exit the relationship, if she knows that the other player will exit in the next day.

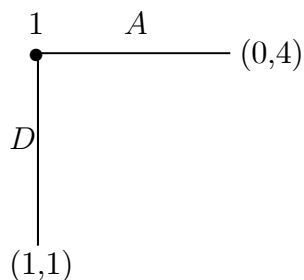


In the third day, Player 1 moves, choosing between going across ( $\alpha$ ) or down ( $\delta$ ). If he goes across, he would get 2; if he goes down, he would get the higher payoff of 3. Hence, according to the procedure, he goes down. Selecting the move  $\delta$  for the node at hand, one reduces the game as follows:



Here, the part of the game that starts at the last decision node is replaced with the payoff vector associated with the selected move,  $\delta$ , of the player at this decision node.

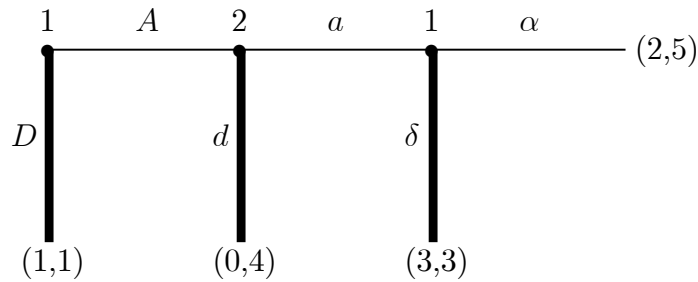
In the second day, Player 2 moves, choosing between going across ( $a$ ) or down ( $d$ ). If she goes across, she get 3; if she goes down, she gets the higher payoff of 4. Hence, according to the procedure, she goes down. Selecting the move  $\delta$  for the node at hand, one reduces the game further as follows:



Once again, the part of the game that starts with the node at hand is replaced with the payoff vector associated with the selected move,  $d$ . Now, Player 1 gets 0 if he goes across ( $A$ ), and gets 1 if he goes down ( $D$ ). Therefore, he goes down. The procedure results in the following strategy profile:

That is, at each node, the player who is to move goes down, exiting the relationship.

Let's go over the assumptions that we have made in constructing this strategy profile.



We assumed that Player 1 will act rationally at the last date, when we reckoned that he goes down. When we reckoned that Player 2 goes down in the second day, we assumed that Player 2 assumes that Player 1 will act rationally on the third day, and also assumed that she is rational, too. On the first day, Player 1 anticipates all these. That is, he is assumed to know that Player 2 is rational, and that she will keep believing that Player 1 will act rationally on the third day.

This example also illustrates another notion associated with backward induction – commitment (or the lack of commitment). Note that the outcomes on the third day (i.e.,  $(3,3)$  and  $(2,5)$ ) are both strictly better than the equilibrium outcome  $(1,0)$ . But they cannot reach these outcomes, because Player 2 cannot commit to go across, and anticipating that Player 2 will go down, Player 1 exits the relationship in the first day. There is also a further commitment problem in this example. If Player 1 were able to commit to go across on the third day, then Player 2 would definitely go across on the second day. In that case, Player 1 would go across on the first. Of course, Player 1 cannot commit to go across on the third day, and the game ends in the first day, yielding the low payoffs  $(1,0)$ .

## 9.2 Backward Induction and Nash Equilibrium

Careful readers must have noticed that the strategy profile resulting from the backward induction above is a Nash equilibrium. (If you have not noticed that, check that it is indeed a Nash equilibrium). This not a coincidence:

**Proposition 9.1** *In a game with finitely many nodes, backward induction always results*

in a Nash equilibrium.

**Proof.** Let  $s^* = (s_1^*, \dots, s_n^*)$  be the outcome of Backward Induction. Consider any player  $i$  and any strategy  $s_i$ . To show that  $s^*$  is a Nash equilibrium, we need to show that

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*),$$

where  $s_{-i}^* = (s_j^*)_{j \neq i}$ . Take any node

- at which player  $i$  moves, and
- $s_i^*$  and  $s_i$  prescribe the same moves for player  $i$  at every node that comes after this node.

(There is always such a node; for example the last node player  $i$  moves.) Consider a new strategy  $s_i^1$  according to which  $i$  plays everywhere according to  $s_i$  except for the above node, where he plays according to  $s_i^*$ . According to  $(s_i^1, s_{-i}^*)$  or  $(s_i, s_{-i}^*)$ , after this node, the play is as in  $(s_i^*, s_{-i}^*)$ , the outcome of the backward induction. Moreover, in the construction of  $s^*$ , we have had selected the best move for player  $i$  given this continuation play. Therefore, the change from  $s_i$  to  $s_i^1$ , which follows the backward induction recommendation, can only increase the payoff of  $i$ :

$$u_i(s_i^1, s_{-i}^*) \geq u_i(s_i, s_{-i}^*).$$

Applying the same procedure to  $s_i^1$ , now construct a new strategy  $s_i^2$  that differs from  $s_i^1$  only at one node, where player  $i$  plays according to  $s_i^*$ , and

$$u_i(s_i^2, s_{-i}^*) \geq u_i(s_i^1, s_{-i}^*).$$

Repeat this procedure, producing a sequence of strategies  $s_i \neq s_i^1 \neq s_i^2 \neq \dots \neq s_i^m \neq \dots$ . Since the game has finitely many nodes, and we are always changing the moves to those of  $s_i^*$ , there is some  $M$  such that  $s_i^M = s_i^*$ . By construction, we have

$$u_i(s^*) = u_i(s_i^M, s_{-i}^*) \geq u_i(s_i^{M-1}, s_{-i}^*) \geq \dots \geq u_i(s_i^1, s_{-i}^*) \geq u_i(s_i, s_{-i}^*),$$

completing the proof. ■

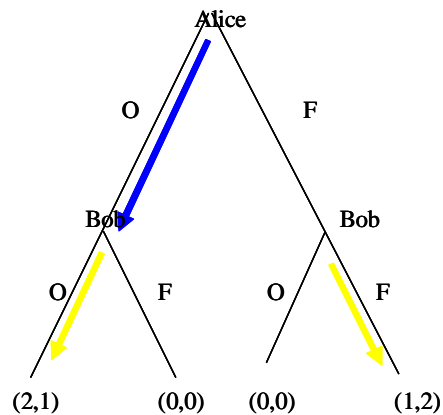
It is tempting to conclude that backward induction results in Nash equilibrium because one plays a best response at every node, finding the above proof unnecessarily long.

Since one takes his future moves given and picks only a move for the node at hand, choosing the best moves at the given nodes does not necessarily lead to a best response among all contingent plans in general.

**Example 9.1** Consider a single player, who chooses between good and bad everyday forever. If he chooses good at everyday, he gets 1, and he gets 0 otherwise. Clearly, the optimal plan is to play good everyday, yielding 1. Now consider the strategy according to which he plays bad everyday at all nodes. This gives him 0. But the strategy satisfies the condition of backward induction (although backward induction cannot be applied to this game with no end node). At any node, according to the moves selected in the future, he gets zero regardless of what he does at the current node.

The above pathological case is a counterexample to the idea that if one is playing a best move at every node, his plan is a best response. The latter idea is a major principle of dynamic optimization, called the Single-Deviation Principle. It applies in most cases except for the pathological cases as above. The above proof shows that the principle applies in games with finitely many moves. Single-Deviation Principle will be the main tool in the analyses of the infinite-horizon games in upcoming chapters. Studying the above proof is recommended.

But not all Nash equilibria can be obtained by backward induction. Consider the following game of the Battle of the Sexes with perfect information, where Alice moves first.



In this game, backward induction leads to the strategy profile identified in the figure, according to which Bob goes wherever Alice goes, and Alice goes to Opera. There is another Nash equilibrium: Alice goes to Football game, and Bob goes to Football game

at both of his decision nodes. Let's see why this is a Nash equilibrium. Alice plays a best response to the strategy of Bob: if she goes to Football she gets 1, and if she goes to Opera she gets 0 (as they do not meet). Bob's strategy (FF) is also a best response to Alice's strategy: under this strategy he gets 2, which is the highest he can get in this game.

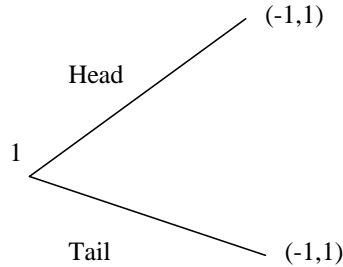
One can, however, discredit the latter Nash equilibrium because it relies on an sequentially irrational move at the node after Alice goes to Opera. This node does not happen according to Alice's strategy, and it is therefore ignored in Nash equilibrium. Nevertheless, if Alice goes to Opera, going to football game would be irrational for Bob, and he would rationally go to Opera as well. And Alice should foresee this and go to Opera. Sometimes, we say that this equilibrium is based on "an incredible threat", with the obvious interpretation.

This example illustrates a shortcoming of the usual rationality condition, which requires that one must play a best response (as a complete contingent plan) at the beginning of the game. While this requires that the player plays a best response at the nodes that he assigns positive probability, it leaves the player free to choose any move at the nodes that he puts zero probability—because all the payoffs after those nodes are multiplied by zero in the expected utility calculation. Since the likelihoods of the nodes are determined as part of the solution, this may lead to somewhat erroneous solutions in which a node is not reached because a player plays irrationally at the node, anticipating that the node will not be reached, as in (Football, FF) equilibrium. Of course, this is erroneous in that when that node is reached the player cannot pretend that the node will not be reached as he will know that the is reached by the definition of information set. Then, he must play a best response taking it given that the node is reached.

### 9.3 Commitment

In this game, Alice can commit to going to a place, but Bob cannot. If we trust the outcome of backward induction, this commitment helps Alice and hurts Bob. (Although the game is symmetric Alice gets a higher payoff.) It is tempting to conclude that ability to commit is always good. While this is true in many games, in some games it is not the case. For example, consider the Matching Pennies with Perfect Information, depicted

in Figure 3.4. Let us apply backward induction. If Player 1 chooses Head, Player 2 will play Head; and if Player 1 chooses Tail, Player 2 will prefer Tail, too. Hence, the game is reduced to



In that case, Player 1 will be indifferent between Head and Tail, choosing any of these two options or any randomization between these two acts will give us an equilibrium with backward induction. In either equilibrium, Player 2 beats Player 1.

## 9.4 Multiple Solutions

This example shows that backward induction can lead to multiple equilibria. Here, in one equilibrium, Player 1 chooses Head, in another one Player 1 chooses Tail, and yet in another mixed strategy equilibrium, he mixes between the two strategies. Each mixture probability corresponds to a different equilibrium. In all of these equilibria, the payoffs are the same. In general, however, backwards induction can lead to multiple equilibria with quite different outcomes.

**Example—Multiple Solutions** Consider the game in Figure 9.2. According to backward induction, in his nodes on the right and at the bottom, Player 1 goes down, choosing  $d$  and  $\delta$ , respectively. This leads to the reduced game in Figure 9.3. Clearly, in the reduced game, both  $x$  and  $y$  yield 2 for Player 2, while  $z$  only yields 1. Hence, she must choose either  $x$  or  $y$  or any randomization between the two. In other words, for any  $p \in [0, 1]$ , the mixed strategy that puts  $p$  on  $x$ ,  $1 - p$  on  $y$  and 0 on  $z$  can be selected by the backward induction. Select such a strategy. Then, the payoff vector associated with



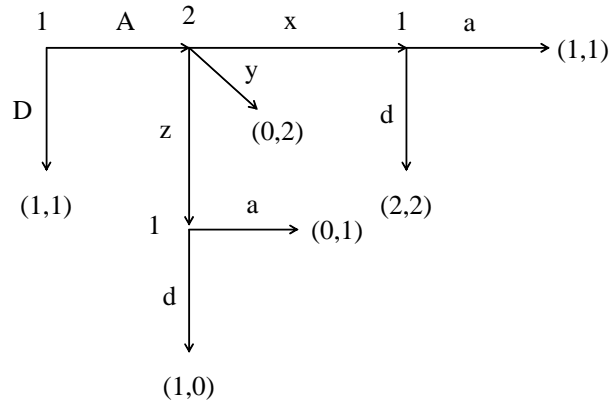


Figure 9.2: A game with multiple backward induction solutions.

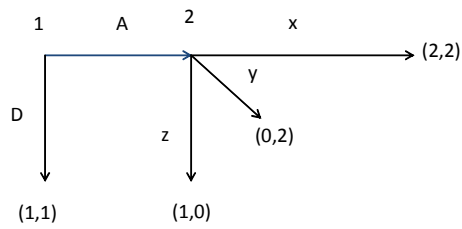
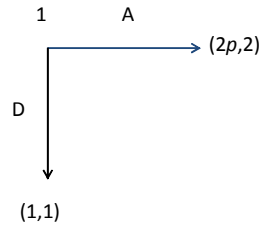


Figure 9.3:

the decision of Player 2 is  $(2p, 2)$ . The game reduces to



The strategy selected for Player 1 depends on the choice of  $p$ . If some  $p > 1/2$  is selected for Player 2, Player 1 must choose  $A$ . This results in the equilibrium in which Player 1 plays *Add* and Player 2 plays  $x$  with probability  $p$  and  $y$  with probability  $1 - p$ . If  $p < 1/2$ , Player 1 must choose  $D$ . In the resulting equilibrium, Player 1 plays *Ddδ* and Player 2 plays  $x$  with probability  $p$  and  $y$  with probability  $1 - p$ . Finally, if  $p = 1/2$  is selected, then Player 1 is indifferent, and we can select any randomization between  $A$  and  $D$ , each resulting in a different equilibrium.

## 9.5 Example—Stackelberg duopoly

In the Cournot duopoly, we assume that the firms set the prices simultaneously. This reflects the assumption that no firm can commit to a quantity level. Sometime a firm may be able to commit to a quantity level. For example, a firm may be already in the market and constructed its factory and warehouses etc, and its production level is fixed. The other firm enters the market later knowing the production level of the first firm. We will consider such a situation, which is called Stackelberg duopoly. There are two firms. The first firm is called the *Leader*, and the second firm is called the *Follower*. As before we take the marginal cost  $c$  constant.

- The Leader first chooses its production level  $q_1$ .
- Then, knowing  $q_1$ , the Follower chooses its own production level  $q_2$ .
- Each firm  $i$  sells its quantity  $q_i$  at the realized market price

$$P(q_1 + q_2) = \max\{1 - (q_1 + q_2), 0\},$$

yielding the payoff of

$$\pi_i(q_1, q_2) = q_i(P(q_1 + q_2) - c).$$

Notice that this defines an extensive form game:

- At the initial node, firm 1 chooses an action  $q_1$ ; the set of allowable actions is  $[0, \infty)$ .
- After each action of firm 1, firm 2 moves and chooses action  $q_2$ ; the set of allowable actions now is again  $[0, \infty)$ .
- Each of these action leads to a terminal node, at which the payoff vector is  $(\pi_1(q_1, q_2), \pi_2(q_1, q_2))$ .

Notice that a strategy of firm 1 is a real number  $q_1$  from  $[0, \infty)$ , and more importantly a strategy of firm 2 is a function from  $[0, \infty)$  to  $[0, \infty)$ , which assigns a production level  $q_2(q_1)$  to each  $q_1$ . These strategies with the utility function  $u_i(q_1, q_2) = \pi_i(q_1, q_2(q_1))$  gives us the normal form.

Let us apply backwards induction. Given  $q_1 \leq 1 - c$ , the best production level for the Follower is

$$q_2^*(q_1) = \frac{1 - q_1 - c}{2},$$

yielding to the payoff vector<sup>2</sup>

$$\begin{pmatrix} \pi_1(q_1, q_2^*(q_1)) \\ \pi_2(q_1, q_2^*(q_1)) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}q_1(1 - q_1 - c) \\ \frac{1}{4}(1 - q_1 - c)^2 \end{pmatrix}. \quad (9.1)$$

By replacing the moves of firm 2 with the associated payoffs we obtain a game in which firm one chooses a quantity level  $q_1$  which leads to the payoff vector in (9.1). In this game firm 1 maximizes  $\frac{1}{2}q_1(1 - q_1 - c)$ , choosing

$$q_1^* = (1 - c)/2,$$

the quantity that it would choose if it were alone.

You should also check that there is also a Nash equilibrium of this game in which the follower produces the Cournot quantity irrespective of what the leader produces, and the leader produces the Cournot quantity. Of course, this is not consistent with backward induction: when the follower knows that the leader has produced the Stackelberg quantity, he will change his mind and produce a lower quantity, the quantity that is computed during the backward induction.

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<sup>2</sup>Note that  $q_1(1 - q_1 - \frac{1 - q_1 - c}{2} - c) = \frac{1}{2}q_1(1 - q_1 - c)$ .

## 9.6 Exercises with Solutions

1. [Midterm 1, 2001] Using Backward Induction, compute an equilibrium of the game in Figure 3.9.

**Solution:** See Figure 9.4.

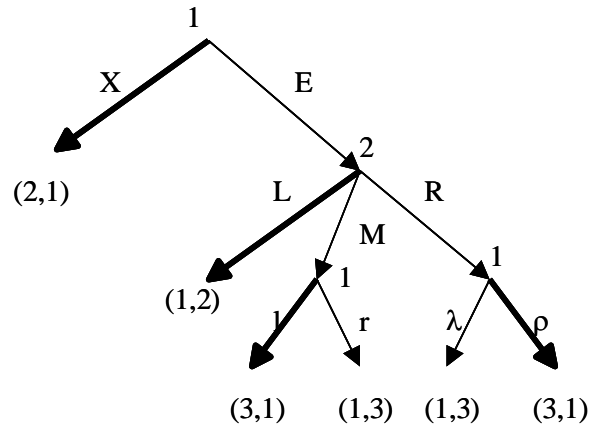


Figure 9.4:

2. Consider the game in Figure 9.5.

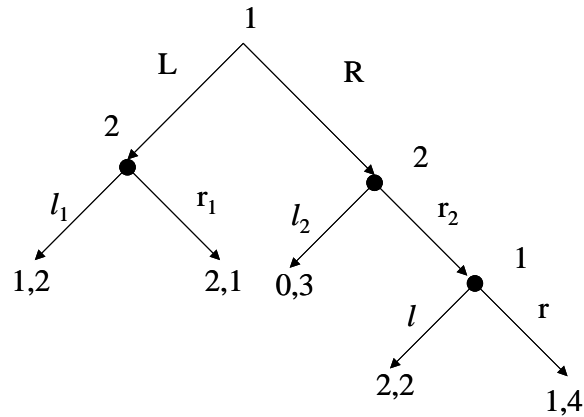


Figure 9.5:

- (a) Apply backward induction in this game. State the rationality/knowledge assumptions necessary for each step in this process.

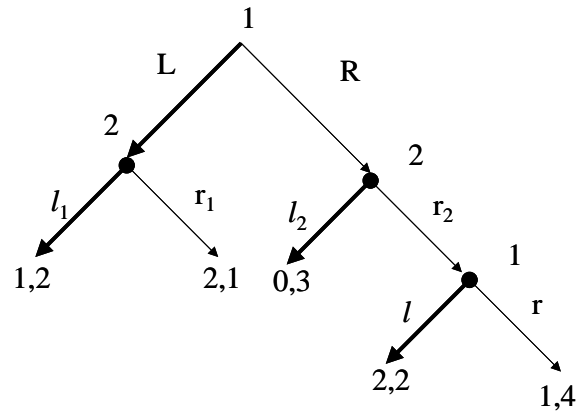


Figure 9.6:

**Solution:** The backward induction outcome is as below. First eliminate action  $r_1$  for Player 2, by assuming that *Player 2 is sequentially rational* and hence will not play  $r_1$ , which is conditionally dominated by  $l_1$ . Also eliminate action  $r$  for Player 1, assuming that *Player 1 is sequentially rational*. This is because  $r$  is conditionally dominated by  $l$ . Second, eliminate  $r_2$ , assuming that *Player 2 is sequentially rational* and that *Player 2 knows that Player 1 will be sequentially rational at future nodes*. This is because, believing that Player 1 will be sequentially rational in the future, Player 2 would believe that Player 1 will not play  $r$ , and hence  $r_2$  would lead to payoff of 2. Being sequentially rational she must play  $l_2$ . Finally, eliminate  $R$ , assuming that (i) *player 1 is sequentially rational*, (ii) *player 1 knows that player 2 is sequentially rational*, and (iii) *player 1 knows that player 2 knows that player 1 will be sequentially rational in the future*. This is because (ii) and (iii) lead Player 1 to conclude that Player 2 will play  $l_1$  and  $l_2$ , and thus by (i) he plays  $L$ . The solution is as in Figure 9.6.

(b) Write this game in normal-form.

**Solution:** Each player has 4 strategies (named by the actions to be chosen).

	$l_1l_2$	$l_1r_2$	$r_1l_2$	$r_1r_2$
$Ll$	1, 2	1, 2	2, 1	2, 1
$Lr$	1, 2	1, 2	2, 1	2, 1
$Rl$	0, 3	2, 2	0, 3	2, 2
$Rr$	0, 3	1, 4	0, 3	1, 4

- (c) Find all the rationalizable strategies in this game —use the normal form. State the rationality/knowledge assumptions necessary for each elimination.

**Solution:** First,  $Rr$  is strictly dominated by the mixed strategy that puts probability .5 on each of  $Ll$  and  $Rl$ . Assuming that *Player 1 is rational*, we conclude that he would not play  $Rr$ . We eliminate  $Rr$ , so the game is reduced to

	$l_1l_2$	$l_1r_2$	$r_1l_2$	$r_1r_2$
$Ll$	1, 2	1, 2	2, 1	2, 1
$Lr$	1, 2	1, 2	2, 1	2, 1
$Rl$	0, 3	2, 2	0, 3	2, 2

Now  $r_1r_2$  is strictly dominated by  $l_1l_2$ . Hence, assuming that (i) *Player 2 is rational*, and that (ii) *Player 2 knows that player 1 is rational*, we eliminate  $r_1r_2$ . This is because, by (ii), Player 2 knows that Player 1 will not play  $Rr$ , and hence by (i) she would not play  $r_1r_2$ . The game is reduced to

	$l_1l_2$	$l_1r_2$	$r_1l_2$
$Ll$	1, 2	1, 2	2, 1
$Lr$	1, 2	1, 2	2, 1
$Rl$	0, 3	2, 2	0, 3

There is no strictly dominated strategy in the remaining game. Therefore, the all the remaining strategies are rationalizable.

- (d) Comparing your answers to parts (a) and (c), briefly discuss whether or how the rationality assumptions for backward induction and rationalizability differ.

**Solution:** Backward induction gives us a much sharper prediction compared to that of rationalizability. This is because the notion of sequential rationality is much stronger than rationality itself.

(e) Find all the Nash equilibria in this game.

**Solution:** The only Nash equilibria are the strategy profiles in which player 1 mixes between the strategies  $Ll$  and  $Lr$ , and 2 mixes between  $l_1l_2$  and  $l_1r_2$ , playing  $l_1l_2$  with higher probability:

$$NE = \{(\sigma_1, \sigma_2) \mid \sigma_1(Ll) + \sigma_1(Lr) = 1, \sigma_2(l_1l_2) + \sigma_2(l_1r_2) = 1, \sigma_2(l_1r_2) \leq 1/2\}.$$

(If you found the pure strategy equilibria (namely,  $(Ll, l_1l_2)$  and  $(Lr, l_1l_2)$ ), you will get most of the points.)

3. [Midterm 1, 2011] A committee of three members, namely  $m = 1, 2, 3$ , is to decide on a new bill that would make file sharing more difficult. The value of the bill to member  $m$  is  $v_m$  where  $v_3 > v_2 > v_1 > 0$ . The music industry, represented by a lobbyist named Alice, stands to gain  $W$  from the passage of the bill, and the file-sharing industry, represented by a lobbyist named Bob, stands to lose  $L$  from the passage of the bill where  $W > L > 0$ . Consider the following game.

- First, Alice promises non-negative contributions  $a_1, a_2$ , and  $a_3$  to the members 1, 2, and 3, respectively, where  $a_m$  is to be paid to member  $m$  by Alice if the bill passes.
- Then, observing  $(a_1, a_2, a_3)$ , Bob promises non-negative contributions  $b_1, b_2$ , and  $b_3$  to the members 1, 2, and 3, respectively, where  $b_m$  is to be paid to member  $m$  by Bob if the bill does not pass.
- Finally, each member  $m$  votes, voting for the bill if  $v_m + a_m > b_m$  and against the bill otherwise. The bill passes if and only if at least two members vote for it.
- The payoff of Alice is  $W - (a_1 + a_2 + a_3)$  if the bill passes and zero otherwise. The payoff of Bob is  $-L$  if the bill passes and  $-(b_1 + b_2 + b_3)$  otherwise.

Assuming that  $2v_3 > L > 2v_2$ , use backward induction to compute a Nash equilibrium of this game. (Note that Alice and Bob are the only players here because

the actions of the committee members are fixed already.) [**Hint:** Bob chooses not to contribute when he is indifferent between contribution and not contributing at all.]

**Solution:** Given any  $(a_1, a_2, a_3)$  by Alice, for each  $m$ , write  $p_m(a_m) = v_m + a_m$  for the "price" of member  $m$  for Bob. If the total price of the cheapest two members exceeds  $L$  (i.e.,  $\sum_m p_m(a_m) - \max_m p_m(a_m) \geq L$ ), then Bob needs to pay at least  $L$  to stop the bill, in which case, he contributes 0 to each member. If the total price of the cheapest two members is lower than  $L$ , then the only best response for Bob is to pay exactly the cheapest two members their price and pay nothing to the remaining member, stopping the bill, which would have cost him  $L$ . In sum, Bob's strategy is given by

$$b_m^*(a_1, a_2, a_3) = \begin{cases} v_m + a_m & \text{if } \sum_{m'} p_{m'}(a_{m'}) - \max_{m'} p_{m'}(a_{m'}) < L \text{ and } m = m^* \\ 0 & \text{otherwise,} \end{cases}$$

where  $m^*$  is the most expensive member, which is chosen randomly when there is a tie.<sup>3</sup>

Given  $b^*$ , as a function of  $(a_1, a_2, a_3)$ , Alice's payoff is

$$U_A(a_1, a_2, a_3) = \begin{cases} W - (a_1 + a_2 + a_3) & \text{if } \sum_m p_m(a_m) - \max_m p_m(a_m) \geq L \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, this is maximized either at some  $(a_1, a_2, a_3)$  with  $\sum_m p_m(a_m) - \max_m p_m(a_m) = L$  (i.e. the cheapest two members costs exactly  $L$  to Bob) or at  $(0, 0, 0)$ . Since  $2v_3 > L > 2v_2$ , Alice can set the prices of 1 and 2 to  $L/2$  by contributing  $(L/2 - v_1, L/2 - v_2, 0)$ , which yields her  $W - L + v_1 + v_2 > 0$  as  $W > L$ . Her strategy is

$$a^* = (L/2 - v_1, L/2 - v_2, 0).$$

**Bonus:** Use backward induction to compute a Nash equilibrium of this game. without assuming  $2v_3 > L > 2v_2$ .

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<sup>3</sup>Those who wrote Bob's strategy wrongly as  $(0, 0, 0)$  or any other vector of numbers will lose 7 points for that. Clearly,  $(0, 0, 0)$  cannot be a strategy for Bob in this game, showing a colossal lack of understanding of the subject.



Answer: First consider the case  $L \leq 2v_3$ . Then, Alice chooses a contribution vector  $(a_1, a_2, 0)$  such that  $a_1 + a_2 + v_1 + v_2 = L$ ,  $a_1 + v_1 \leq v_3$ , and  $a_2 + v_2 \leq v_3$ . Such a vector is feasible because  $L < 2v_3$  and  $v_3 > v_2 > v_1 > 0$ . Optimality of this contribution is as before.

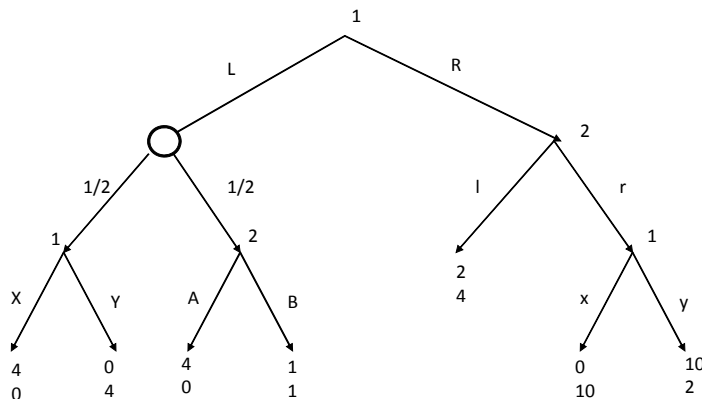
Now consider the case  $L > 2v_3$ . Now, Alice must contribute to all members in order to pass the bill, and the optimality requires that the prices of all members are  $L/2$  (as Bob buys the cheapest two). That is, she must contribute

$$a^{**} = (L/2 - v_1, L/2 - v_2, L/2 - v_3).$$

Since this costs Alice  $3L/2 - (v_1 + v_2 + v_3)$ , she makes such a contribution to pass the bill if and only if  $3L/2 \leq W + (v_1 + v_2 + v_3)$ . Otherwise, she contributes  $(0, 0, 0)$  and the bill fails.

## 9.7 Exercises

1. In Stackelberg duopoly example, for every  $q_1 \in (0, 1)$ , find a Nash equilibrium in which Firm 1 plays  $q_1$ .
2. Apply backward induction to the "sequential Stackelberg oligopoly" with  $n$  firms: Firm 1 chooses  $q_1$  first, firm 2 chooses  $q_2$  second, firm 3 chooses  $q_3$  third, ..., and firm  $n$  chooses  $q_n$  last.
3. [Homework 2, 2011] Use backward induction to compute a Nash equilibrium of the following game.



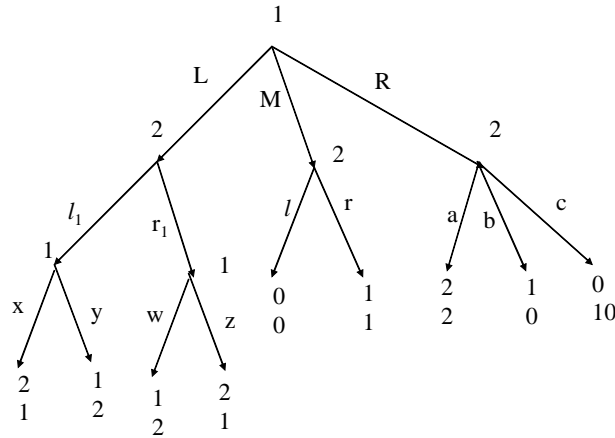


Figure 9.7:

4. [Homework 2, 2002] Apply backward induction in the game of Figure 9.7.
5. [Homework 2, 2002] Three gangsters armed with pistols, Al, Bob, and Curly, are in a room with a suitcase of money. Al, Bob, and Curly have 20%, 40% and 70% chances of killing their target, respectively. Each has one bullet. First Al shoots targeting one of the other two gangster. After Al, if alive, Bob shoots, targeting one of the surviving gangsters. Finally, if alive, Curly shoots, targeting again one of the surviving gangsters. The survivors split the money equally. Find a subgame-perfect equilibrium.
6. [Midterm 1 Make Up, 2001] Find all pure-strategy Nash equilibria in Figure 9.8. Which of these equilibria are can be obtained by backward induction?
7. [Final Make up, 2000] Find the subgame-perfect equilibrium of the following 2-person game. First, player 1 picks an integer  $x_0$  with  $1 \leq x_0 \leq 10$ . Then, player 2 picks an integer  $y_1$  with  $x_0 + 1 \leq y_1 \leq x_0 + 10$ . Then, player 1 picks an integer  $x_2$  with  $y_1 + 1 \leq x_2 \leq y_1 + 10$ . In this fashion, they pick integers, alternatively. At each time, the player moves picks an integer, by adding an integer between 1 and 10 to the number picked by the other player last time. Whoever picks 100 wins the game and gets 100; the other loses the game and gets zero.
8. [Final, 2001] Consider the extensive form game in Figure 9.9.

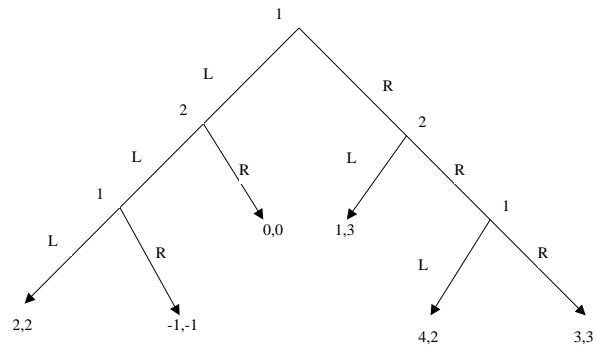


Figure 9.8:

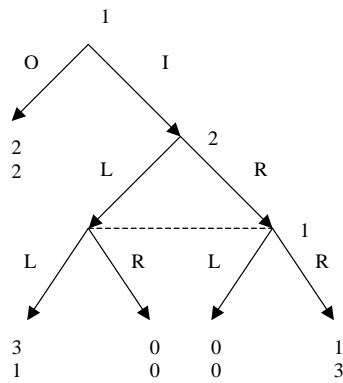


Figure 9.9:

- (a) Find the normal form representation of this game.
- (b) Find all rationalizable pure strategies.
- (c) Find all pure strategy Nash equilibria.
- (d) Which strategies are consistent with all of the following assumptions?
- (i) 1 is rational.
  - (ii) 2 is sequentially rational.
  - (iii) at the node she moves, 2 knows (i).
  - (iv) 1 knows (ii) and (iii).
9. [Final 2004] Use backward induction to find a Nash equilibrium for the following game, which is a simplified version of a game called Weakest Link. There are 4 risk-neutral contestants, 1, 2, 3, and 4, with "values"  $v_1, \dots, v_4$  where  $v_1 > v_2 > v_3 > v_4 > 0$ . Game has 3 rounds. At each round, an outside party adds the value of each "surviving" contestant to a common account,<sup>4</sup> and at the end of third round one of the contestants wins and gets the amount collected in the common account. We call a contestant surviving at a round if he was not eliminated at a previous round. At the end of rounds 1 and 2, the surviving contestants vote out one of the contestants. The contestants vote sequentially in the order of their indices (i.e., 1 votes before 2; 2 votes before 3, and so on), observing the previous votes. The contestant who gets the highest vote is eliminated; the ties are broken at random. At the end of the third round, a contestant  $i$  wins the contest with probability  $v_i / (v_i + v_j)$ , where  $i$  and  $j$  are the surviving contestants at the third round. (Be sure to specify which player will be eliminated for each combination of surviving contestants, but you need not necessarily specify how every contestant will vote at all contingencies.)
10. [Midterm 1, 2011] A committee of three members, namely  $m = 1, 2, 3$ , is to decide on a new bill that would make file sharing more difficult. The value of the bill to member  $m$  is  $v_m$  where  $v_3 > v_2 > v_1 > 0$ . The music industry, represented by

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<sup>4</sup>For example, if contestant 2 is eliminated in the first round and contestant 4 is eliminated in the second round, the total amount in the account is  $(v_1 + v_2 + v_3 + v_4) + (v_1 + v_3 + v_4) + (v_1 + v_3)$  at the end of the game.

a lobbyist named Alice, stands to gain  $W$  from the passage of the bill, and the file-sharing industry, represented by a lobbyist named Bob, stands to lose  $L$  from the passage of the bill where  $W > L > 0$ . Consider the following game.

- First, Alice promises non-negative contributions  $a_1, a_2$ , and  $a_3$  to the members 1, 2, and 3, respectively, where  $a_m$  is to be paid to member  $m$  by Alice if the bill passes.
- Then, observing  $(a_1, a_2, a_3)$ , Bob promises non-negative contributions  $b_1, b_2$ , and  $b_3$  to the members 1, 2, and 3, respectively, where  $b_m$  is to be paid to member  $m$  by Bob if the bill does not pass.
- Finally, each member  $m$  votes, voting for the bill if  $v_m + a_m > b_m$  and against the bill otherwise. The bill passes if and only if at least two members vote for it.
- The payoff of Alice is  $W - (a_1 + a_2 + a_3)$  if the bill passes and zero otherwise. The payoff of Bob is  $-L$  if the bill passes and  $-(b_1 + b_2 + b_3)$  otherwise.

Use backward induction to compute a Nash equilibrium of this game. (Note that Alice and Bob are the only players here because the actions of the committee members are fixed already.)



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Fall 2012

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