Lectures 17-19
Static Applications with Incomplete Information

14.12 Game Theory

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Road Map

1. Cournot Duopoly
2. First Price Auction
    1. Linear Symmetric Equilibrium
    2. Symmetric Equilibrium
3. Double Auction/Bargaining
4. Coordination with incomplete information

Recall: Bayesian Game & Bayesian Nash Equilibrium

A Bayesian game is a list

$$G = \{A_1,\ldots,A_n;T_1,\ldots,T_n;p_1,\ldots,p_n;u_1,\ldots,u_n\}$$

where

- A_i is the action space of i (a_i in A_i)
- T_i is the type space of i (t_i)
- $p_i(t_{-i}|t_i)$ is i's belief about the other players
- $u_i(a_1,\ldots,a_n;t_1,\ldots,t_n)$ is i's payoff.

A strategy profile $s^* = (s_1^*,\ldots, s_1^*)$ is a Bayesian Nash equilibrium iff $s_i^*(t_i)$ is a best response to s_{-i}^* for each t_i.

An Example

	L	R
X	θ,γ	1,2
Y	$-1,\gamma$	$\theta,0$

- $\theta \in \{0,2\}$, known by Player 1
- $\gamma \in \{1,3\}$, known by Player 2
- All values are equally likely
- $T_1 = \{0,2\}; T_2 = \{1,3\}$
- $p(\mathrm{t}) = p_i(t_j | t_i) = 1/2$
- $A_1 = \{X,Y\}$; $A_2 = \{L,R\}$

A Bayesian Nash Equilibrium:

- $s_1(0) = X$
- $s_1(2) = X$
- $s_2(1) = R$
- $s_2(3) = L$

Linear Cournot Duopoly

- Two firms, 1 & 2; $P = 1-(q_1+q_2)$
- Marginal cost of 1: $c_1 = 0$, common knowledge
- Marginal cost of 2: c_2, privately known by 2

$$c_2 = \begin{cases} c_H \text{ with pr } \theta \\ c_L \text{ with pr } 1-\theta \end{cases}$$

BNE in LCD

- q_1^*, $q_2^*(c_H)$, $q_2^*(c_L)$
- 1 plays best reply:

$$q_1^* = (1-[\theta q_2^*(c_H) + (1-\theta) q_2^*(c_L)])/2$$

- 2 plays best reply at c_H:

$$q_2^*(c_H) = (1- q_1^* - c_H)/2$$

- 2 plays best reply at c_L:

$$q_2^*(c_L) = (1- q_1^* - c_L)/2$$

Solution

$$q_1^* = \frac{1 + \theta c_H + (1-\theta)c_L}{3}$$

$$q_2^*(c_H) = \frac{1-2c_H}{3} + (1-\theta)\frac{c_H - c_L}{6}$$

$$q_2^*(c_L) = \frac{1-2c_L}{3} - \theta\frac{c_H - c_L}{6}$$

First price auction

- Two bidders, 1 & 2, and an object
- v_i = value of object for bidder *i*, privately known by *i*
- v_i ~ iid with Uniform [0,1]
- Each *i* bids b_i, simultaneously, and the highest bidder buys, paying his own bid

First Price Auction – Game

- T_i =
- $p_i(.|v_i)$ =
- A_i =
- Payoffs:

$$u_i(b_1, b_2, v_1, v_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ (v_i - b_i)/2 & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

Symmetric, Linear BNE

1. Assume a symmetric "linear" BNE:

$$b_1(v_1) = a + cv_1$$

$$b_2(v_2) = a + cv_2$$

2. Compute best reply function of each type:

$$b_i = (a + v_i)/2$$

3. Verify that best reply functions are affine:

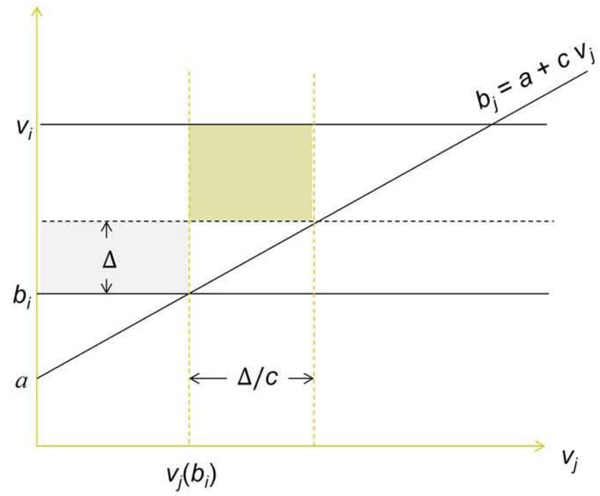
$$b_i(v_i) = a/2 + (1/2)v_i$$

4. Compute the constants a and c:

$$a = a/2 \;\&\; c = 1/2$$

$$a = 0;\; c = 1/2$$

Payoff from bid & its change



Any symmetric BNE

1. Assume a symmetric BNE (of the form):

$$b_1(v_1) = b(v_1)$$
$$b_2(v_2) = b(v_2)$$

2. Compute the (1st-order condition for) best reply of each type:

$$-b^{-1}(b_i^*) + (v_i - b_i^*)\left.\frac{db^{-1}}{db_i}\right|_{b_i = b_i^*} = 0$$

3. Identify best reply with BNE action: $b_i^* = b(v_i)$
4. Substitute 3 in 2:

$$-v_i b'(v_i) + (v_i - b(v_i)) = 0$$

5. Solve the differential equation (if possible):

$$b(v_i) = v_i/2$$

Double Auction

- Players: A Seller & A Buyer
- Seller owns an object, whose value
  - for Seller is v_S, privately known by Seller
  - for the buyer is v_B, privately known by Buyer
  - v_S and v_B are iid with uniform on [0,1]
- Buyer and Seller post p_B and p_S
- If $p_B \geq p_S$, Buyer buys the object at price

$$p = (p_B + p_S)/2$$

- There is no trade otherwise.

Double Auction – Game

- T_i =
- $p_i(.|v_i)$ =
- A_i =
- Payoffs:

$$u_B(p_B, p_S, v_B, v_S) = \begin{cases} v_B - (p_B + p_S)/2 & p_B \geq p_S \\ 0 & otherwise \end{cases}$$

$$u_S(p_B, p_S, v_B, v_S) = \begin{cases} (p_B + p_S)/2 - v_S & p_B \geq p_S \\ 0 & otherwise \end{cases}$$

A BNE

$$p_B(v_B) = \begin{cases} X & \text{if } v_B \geq X \\ 0 & \text{otherwise} \end{cases} \qquad p_S(v_S) = \begin{cases} X & \text{if } v_S \leq X \\ 1 & \text{otherwise} \end{cases}$$

Linear BNE

1. Assume a "linear" BNE:

$$p_B(v_B) = a_B + c_B v_B$$
$$p_S(v_S) = a_S + c_S v_S$$

2. Compute best reply function of each type:

$$p_B = (2/3)\ v_B + a_S/3$$
$$p_S = (2/3)\ v_S + (a_B + c_B)/3.$$

3. Verify that best reply functions are affine
4. Compute the constants:

$$c_B = c_S = 2/3;\ a_B = a_S/3\ \&\ a_S = (a_B + c_B)/3$$
$$a_B\ = 1/12;\ a_S = 1/4$$

Computing Best Replies

$$E[u_B \mid v_B] = \int_0^{\frac{p_B - a_S}{c_S}} \left[v_B - \frac{p_B + a_S + c_S v_S}{2} \right] dv_S$$

1st order condition ($\partial E[u_B|v_B]/\partial p_B = 0$):

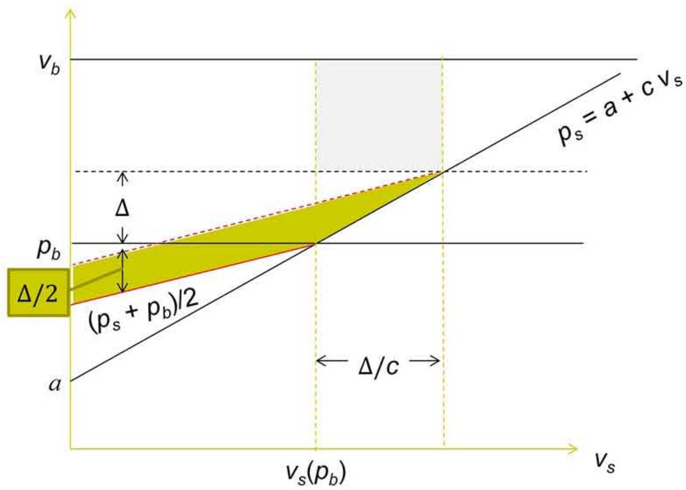
$$\frac{1}{c_S}(v_B - p_B) - \frac{p_B - a_S}{2c_S} = 0$$

$$E[u_S \mid v_S] = \int_{\frac{p_S - a_B}{c_B}}^{1} \left[\frac{p_S + a_B + c_B v_B}{2} - v_S \right] dv_B$$

1st order condition ($\partial E[u_S|v_S]/\partial p_S = 0$):

$$-\frac{1}{c_B}(p_S - v_S) + \frac{1}{2}\left(1 - \frac{p_S - a_B}{c_B}\right) = 0$$

Payoff from bid & itc change



Trade in linear BNE

- $p_B = (2/3)\ v_B + 1/12$
- $p_S = (2/3)\ v_S + 1/4$.
- Trade $\Leftrightarrow$ $p_B \geq p_S$
- $\Leftrightarrow$ $v_B - v_S \geq$ ¼.

Coordination with incomplete information

- Coordination is an important problem
  - Bank runs
  - Currency attacks
  - Investment in capital and human capital
  - R&D and Marketing departments
  - Development
- With complete information, multiple equilibria
- With incomplete information, unique equilibrium

A simple partnership game

	Invest	NotInvest
Invest	θ,θ	$\theta - 1, 0$
NotInvest	$0,\theta - 1$	$0,0$

θ is common knowledge

$\theta < 0$

	Invest	NotInvest
Invest	θ,θ	$\theta - 1, 0$
NotInvest	$0,\theta - 1$	$0,0$

θ is common knowledge

$\theta > 1$

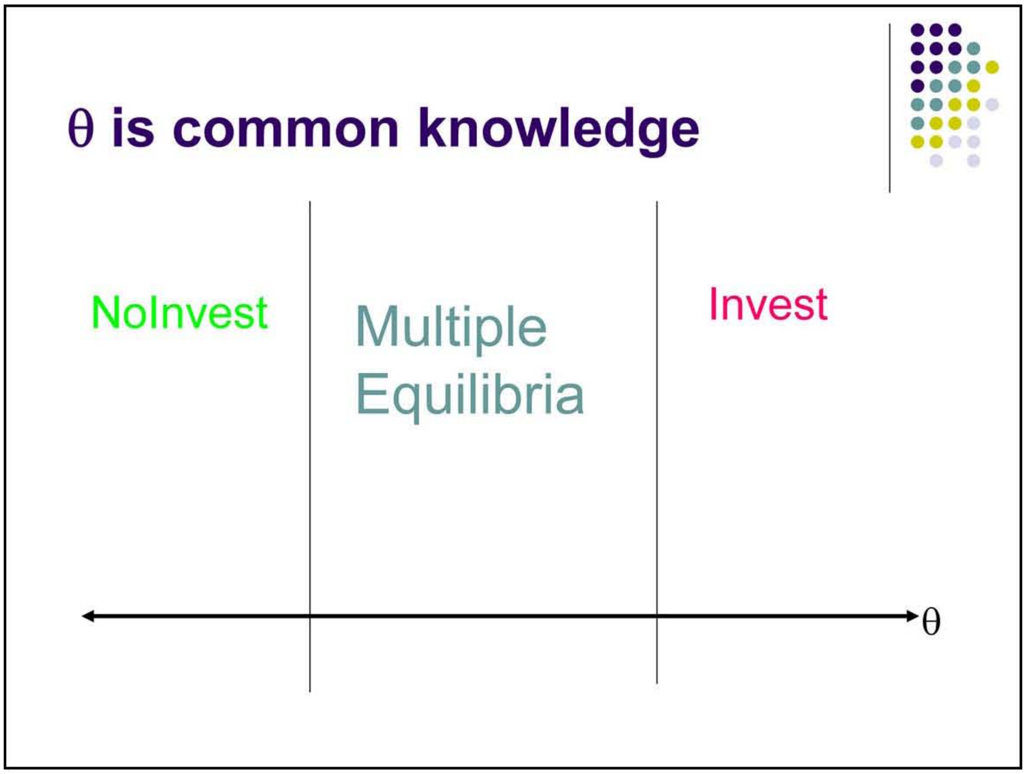
	Invest	NotInvest
Invest	θ,θ	$\theta - 1, 0$
NotInvest	$0,\theta-1$	$0,0$

θ is common knowledge

$0 < \theta < 1$

Multiple Equilibria!!!

	Invest	NotInvest
Invest	θ, θ	$\theta - 1, 0$
NotInvest	$0, \theta - 1$	$0, 0$


θ is common knowledge
NoInvest
Multiple Equilibria
Invest
θ

θ is not common knowledge

- θ is uniformly distributed over a very, very large interval
- Each player i gets a signal

$$x_i = \theta + \varepsilon\eta_i$$

- (η_1, η_2) iid with uniform on [-1,1]; $\varepsilon>0$ small
- The distribution is common knowledge.
- Note: $\Pr(x_j < x_i \mid x_i) = \Pr(x_j > x_i \mid x_i) = 1/2$

Payoffs and best response

	Invest	NotInvest
Invest	x_1, x_2	x_1-1, 0
NotInvest	0, x_2-1	0,0

Payoff from Invest = x_i – Pr(NotInvest | x_i)

Payoff from NotInvest = 0

Invest ⇔ $x_i \geq$ Pr(NotInvest | x_i)

Symmetric Monotone BNE

- There is a cutoff x^* s.t.

$$s_i^*(x_i) = \begin{cases} Invest & \text{if } x_i \geq x^* \\ NotInvest & \text{if } x_i < x^* \end{cases}$$

- For $x_i > x^*$,

  $x_i \geq \Pr(s_j^*(x_j)=\text{NotInvest}|x_i) = \Pr(x_j < x^*| x_i)$
- For $x_i < x^*$, $x_i \leq \Pr(x_j < x^*| x_i)$
- By continuity,

$$x^* = \Pr(x_j < x^*| x^*) = ½$$

Unique equilibrium!!!

Risk-dominance

- In a 2 x 2 game, a strategy is said to be "risk dominant" iff it is a best reply when the other player plays each strategy with equal probabilities.

	Invest	NotInvest
Invest	θ,θ	$\theta-1, 0$
NotInvest	$0,\theta-1$	$0,0$

Invest is RD iff

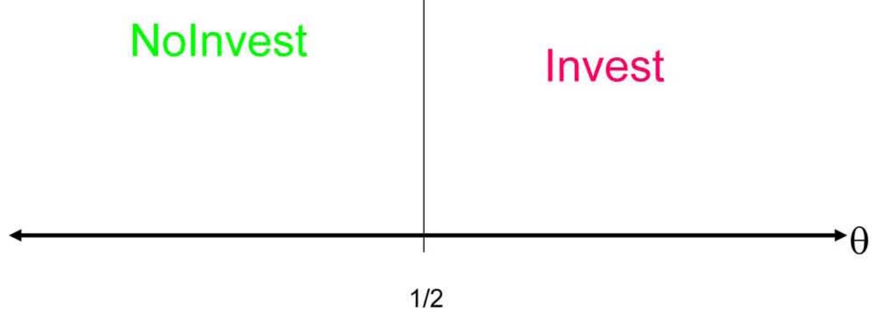
$$0.5\theta + 0.5(\theta-1) > 0$$

$$\Leftrightarrow \theta > 1/2$$

θ is **not** common knowledge but the noise is very small

It is very likely that

risk-dominant strategy is played!!



14.12 Economic Applications of Game Theory
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