Problem 1

Throughout this problem, note that conditional preference is well-defined, by P2.

(a) True. I show that B_1 and B_2 are null if and only if $B_1 \cup B_2$ is null. The result follows by induction on n.

Suppose that B_1 and B_2 are null. Then, for all $f, g, h \in F$,

$$f^h_{|B_1 \cup B_2} = f^{f^h_{|B_2}}_{|B_1} \sim g^{f^h_{|B_2}}_{|B_1} = \left(f^g_{B_2 \setminus B_1}\right)^{g^h_{|B_1}}_{|B_2} \sim g^{g^h_{|B_1}}_{|B_2} = g^h_{|B_1 \cup B_2},$$

where the first $\sim$ follows because B_1 is null and the second $\sim$ follows because B_2 is null. By definition, this implies that $B_1 \cup B_2$ is null.

Suppose that $B_1 \cup B_2$ are null. Then, for all $f, g, h \in F$,

$$f^h_{|B_1} = \left(f^h_{|B_1}\right)^h_{|B_1 \cup B_2} \sim \left(g^h_{|B_1}\right)^h_{|B_1 \cup B_2} = g^h_{|B_1}.$$

By definition, this implies that B_1 is null. Similarly, B_2 is null.

(b) True. Since C is finite, there exists a finite partition $\{D^1, \ldots, D^n\}$ of S such that, if $s, s' \in D^i$ for some $i \in \{1, \ldots, n\}$, then $f(s) = f(s')$ and $g(s) = g(s')$. Let $C^0 = \emptyset$ and let $C^i = \bigcup_{k=1}^i D^k$. For any $i \in \{1, \ldots, n\}$, let f_x be the constant act that always yields consequence $x \equiv f(s)$ for some $s \in D^i$ and let f_y be the constant act that always yields consequence $y \equiv g(s)$ for some $s \in D^i$. If D^i is null then $(f_x)^{f^g_{|C^i}}_{|D^i} \succsim (f_y)^{f^g_{|C^i}}_{|D^i}$ trivially, and if D^i is non-null then $(f_x)^{f^g_{|C^i}}_{|D^i} \succsim (f_y)^{f^g_{|C^i}}_{|D^i}$ by P3. Hence, for every $i \in \{1, \ldots, n\}$,

$$f^g_{|C^i} = (f_x)^{f^g_{|C^i}}_{|D^i} \succsim (f_y)^{f^g_{|C^i}}_{|D^i} = f^g_{|C^{i-1}}.$$

Therefore, $f = f^g_{|S} \succsim f^g_{|\emptyset} = g$.

(c) False. Let $S = \{s, t\}$, $C = \{x, x', y, y'\}$. I denote an act $f : S \to C$ by a pair (a, b), with the interpretation that $f(s) = a$ and $f(t) = b$. Consider the preference relation $\succsim$ given by

$$\begin{aligned}
(x,x) &\succ (x,x') \succ (x',x) \succ (x',x') \succ (x,y) \\
&\succ (x,y') \succ (x',y) \succ (x',y') \succ (y,x) \\
&\succ (y,x') \succ (y',x) \succ (y',x') \succ (y,y) \\
&\succ (y',y) \succ (y,y') \succ (y',y').
\end{aligned}$$

One can check that $\succsim$ satisfies P1-P3, but $x \succ x'$, $y \succ y'$, $f^{x,x'}_{\{s\}} \succ f^{x,x'}_{\{t\}}$, and $f^{y,y'}_{\{t\}} \succ f^{y,y'}_{\{s\}}$.

Problem 2

(a) True. By P1-P5, $\dot{\succsim}$ is a qualitative probability. In what follows, "property 2" and "property 3" refer to properties 2 and 3 of qualitative probability, on page 85 in the lecture notes.

I first prove the result in the special case where $B_1 \cap B_2 = \emptyset$. Note that

$$\begin{aligned} A_2 \cup (A_1 \backslash B_2) &\;\dot{\succsim}\; B_2 \cup (A_1 \backslash B_2) \text{ (by } A_2 \dot{\succsim} B_2,\ A_1 \cap A_2 = \emptyset, \text{ and property 2)} \\ &= A_1 \cup B_2 \\ &= A_1 \cup (B_2 \backslash A_1) \\ &\;\dot{\succsim}\; B_1 \cup (B_2 \backslash A_1) \text{ (by } A_1 \dot{\succsim} B_1,\ B_1 \cap B_2 = \emptyset, \text{ and property 2).} \end{aligned}$$

Since $(A_2 \cup (A_1 \backslash B_2)) \cap (A_1 \cap B_2) = \emptyset$ (by $A_1 \cap A_2 = \emptyset$) and $(B_1 \cup (B_2 \backslash A_1)) \cap (A_1 \cap B_2) = \emptyset$ (by $B_1 \cap B_2 = \emptyset$), this implies that (using property 2 again)

$$A_1 \cup A_2 = (A_2 \cup (A_1 \backslash B_2)) \cup (A_1 \cap B_2) \,\dot{\succsim}\, (B_1 \cup (B_2 \backslash A_1)) \cup (A_1 \cap B_2) = B_1 \cup B_2.$$

Now suppose that $B_1 \cap B_2 \neq \emptyset$. Note that $B_2 \dot{\succsim} B_2 \backslash B_1$, because $B_1 \cap B_2 \dot{\succsim} \emptyset$ (by property 3), and therefore $B_2 = (B_1 \cap B_2) \cup (B_2 \backslash B_1) \,\dot{\succsim}\, \emptyset \cup (B_2 \backslash B_1) = (B_2 \backslash B_1)$ (by property 2). Therefore, $A_2 \dot{\succsim} B_2 \backslash B_1$, so the fact that the result holds in the special case where $B_1 \cap B_2 = \emptyset$ implies that $A_1 \cup A_2 \dot{\succsim} B_1 \cup (B_2 \backslash B_1) = B_1 \cup B_2$.

(b) False. For the simplest counterexample, let $D = \emptyset$, in which case clearly $A \dot{\succsim} B$ for all $A, B \subseteq S$, and in particular $\emptyset \dot{\succsim} S$, which contradicts property 3 of qualitative probability. One can derive a similar contradiction for any null event D, and one can show that $\dot{\succsim}$ given D is indeed a qualitative probability if D is non-null.

(c) True. Note that, by property 1 of a probability measure (also on page 85 in the lecture notes), $p(\emptyset) = 0$ (because $p(\emptyset) = p(\emptyset \cup \emptyset) = 2p(\emptyset)$).

If $A \subseteq S$ is null, then

$$f^{x,x'}_{A} = (f_x)^{x'}_{|A} \sim (f_{x'})^{x'}_{|A} = f^{x,x'}_{\emptyset},$$

and therefore $A \dot{\sim} \emptyset$. Hence, if p represents $\dot{\succsim}$, it follows that $p(A) = p(\emptyset) = 0$.

Conversely, if $p(A) = 0$, then $p(A) = p(\emptyset)$, so if p represents $\dot{\succsim}$ then $A \dot{\sim} \emptyset$. Therefore, $f_A^{x,x'} = (f_x)_{|A}^{x'} \sim (f_{x'})_{|A}^{x'} = f_{\emptyset}^{x,x'}$. But if A were not null, then P3 would imply that $(f_x)_{|A}^{x'} \succ (f_{x'})_{|A}^{x'}$, so it must be that A is null.

Problem 3

(Thanks to Hongkai Zhang, whose solution I used as a model for this)

As per Muhamet's hint, each player has a well-defined continuation value at every history, and she accepts her opponent's proposal if and only if it gives her expected utility at least as great as her continuation value. Therefore, at every history the proposer proposes a Pareto efficient allocation that gives her opponent exactly her continuation value. Due to CARA utility and normally distributed payoffs, the formula for efficient risk-sharing derived in lecture implies that player i's time-t proposal is

$$\begin{aligned} x_i^t &= \frac{\alpha_{-i}}{\alpha_i + \alpha_{-i}} \sum_{s=t}^{T-1} (x_{i,s} + x_{-i,s}) + \sum_{s=0}^{t-1} x_{i,s} + \tau_t \\ x_{-i}^t &= \frac{\alpha_i}{\alpha_i + \alpha_{-i}} \sum_{s=t}^{T-1} (x_{i,s} + x_{-i,s}) + \sum_{s=0}^{t-1} x_{-i,s} - \tau_t \end{aligned}$$

(noting that $\frac{\alpha_{-i}}{\alpha_i + \alpha_{-i}} = \frac{1/\alpha_i}{1/\alpha_i + 1/\alpha_{-i}}$).

It remains only to determine τ_t (and to verity that the fixed component of x_i^t does indeed depend on the history only through t). Note that τ_t is determined by the condition that player $-i$ is indifferent between the assets x_{-i}^t and x_i^{t+1} at time t. Using certainty equivalents, this condition is

$$\begin{aligned} & CE_{-i}\left(\frac{\alpha_i}{\alpha_i + \alpha_{-i}} \sum_{s=t}^{T-1} (x_{i,s} + x_{-i,s}) + \sum_{s=0}^{t-1} x_{-i,s} - \tau_t \right) \\ & \qquad = CE_{-i}\left(\frac{\alpha_i}{\alpha_i + \alpha_{-i}} \sum_{s=t+1}^{T-1} (x_{i,s} + x_{-i,s}) + \sum_{s=0}^{t} x_{-i,s} + \tau_{t+1} \right), \end{aligned}$$

or equivalently

$$CE_{-i}\left(\frac{\alpha_i}{\alpha_i + \alpha_{-i}} (x_{i,t} + x_{-i,t}) - \tau_t \right) = CE_{-i}\left(x_{-i,t} + \tau_{t+1} \right). \tag{1}$$

Recall the following two facts:

1. If X and Y are iid random variables distributed $N(0,\sigma^2)$, then $k(X+Y)$ is distributed $N(0,2k^2\sigma^2)$.

2. With CARA utility, if X is distributed $N(\mu,\gamma^2)$, then $CE(X)=\mu-\frac{\alpha\gamma^2}{2}$.

Using these facts, (1) may be rewritten as

$$-\left(\frac{\alpha_i}{\alpha_i+\alpha_{-i}}\right)^2\alpha_{-i}\sigma^2-\tau_t=-\frac{\alpha_{-i}\sigma^2}{2}+\tau_{t+1},$$

or equivalently

$$\tau_t=\left(\frac{1}{2}-\left(\frac{\alpha_i}{\alpha_i+\alpha_{-i}}\right)^2\right)\alpha_{-i}\sigma^2-\tau_{t+1}. \tag{2}$$

Noting that $\tau_T=0$, (2) determines τ_t for all $t\in\{0,\ldots,T-1\}$ (where $i=1$ if T is odd, and $i=2$ if T is even). (2) can be written more concisely as follows:

$$\begin{aligned}\tau_t &= \left(\frac{1}{2}-\left(\frac{\alpha_i}{\alpha_i+\alpha_{-i}}\right)^2\right)\alpha_{-i}\sigma^2-\tau_{t+1} \text{ if } t \text{ is even}\\ \tau_t &= \left(\frac{T-t+1}{2}\right)\left(\frac{1}{2}+\frac{\alpha_i\alpha_{-i}}{(\alpha_i+\alpha_{-i})^2}\right)(\alpha_{-i}-\alpha_i)\,\sigma^2 \text{ if } t \text{ is odd.}\end{aligned}$$