14.124 Problem Set 3 - **Solutions**

Question 1

(a) Here is the program:

$$
\begin{aligned}
\max_{e_A, e_B, \alpha, \beta} \quad & p[(1-\alpha)e_A - \beta] - (1-p)(\alpha e_B - \beta); \\
s.t. \quad & e_A \in \text{argmax}_{\tilde{e}_A} \alpha \tilde{e}_A - \tilde{e}_A^2; \\
& e_B \in \text{argmax}_{\tilde{e}_B} \alpha \tilde{e}_B - a\tilde{e}_B - b\tilde{e}_B^2; \\
& p(\alpha e_A + \beta - e_A^2) + (1-p)(\alpha e_B + \beta - a e_B - b e_B^2) \geq 0.
\end{aligned}
$$

The first constraint is the agent's IC constraint for task A, the second his IC constraint for task B, and the third his participation constraint.

(b) Clearly, $e_A = \alpha/2$, $e_B = \max\{0, (\alpha - a)/(2b)\}$, and $\beta = -\frac{p}{4}\alpha^2 - \frac{1-p}{4b}(\max\{\alpha - a, 0\})^2$. Therefore, when $\alpha \geq a$, the principal's expected payoff is

$$
\frac{p}{2}\alpha(1-\alpha) - \frac{(1-p)(\alpha - a)\alpha}{2b} + \frac{p}{4}\alpha^2 + \frac{1-p}{4b}(\alpha - a)^2.
$$

The optimal α is thus

$$
\alpha^* = \frac{pb}{pb + 1 - p}.
$$

When $\alpha < a$, $e_B = 0$, and the principal's payoff is

$$\frac{p}{2}\alpha(1-\alpha) + \frac{p}{4}\alpha^2.$$

The optimal value of α is one. When $\frac{pb}{pb+1-p} < a < 1$, the principal's payoff is increasing in α when $\alpha \geq a$ and decreasing in α when $\alpha \geq a$. Therefore, the optimal α is a. To sum up,

$$\alpha^* = \begin{cases} \frac{pb}{pb+1-p}, & \text{if } a \leq \frac{pb}{pb+1-p}; \\ a, & \text{if } \frac{pb}{pb+1-p} < a < 1; \\ 1, & \text{if } a \geq 1. \end{cases}$$

(c) Assume that B is ruled out. Consider the following contract: $\alpha = 1$ and β is the minimum salary that satisfies the agent's participation constraint. Then $e_A = 1/2$ and $e_B = \max\{a/(2b), 0\}$, which are first-best effort levels. In other words, there is no efficiency loss in this contract. Also, the agent's participation constraint holds with equality, so this is the optimal contract for the principal and beats any contracts that satisfy the agent's participation constraint. Therefore, it is always optimal to rule out task B.

Question 2

(a) For each type, $w_i = \theta_i h_i$, and thus the profit is $h_i/\theta_i - \theta_i h_i$. SInce $\theta_i < 1$, the optimal choice of h_i is one, and $w_i = \theta_i$.

This is not implementable when types are not observed, because type L prefers to receive $w_H = \theta_H$ instead of $w_L = \theta_L$.

(b) There is no uncertainty from the agent's perspective, so the agent has effective payoff $w - \theta_i h$. Here is the program:

$$\begin{aligned} \max_{w_i,h_i} \quad & \sum_i p_i(h_i/\theta_i - w_i); \\ s.t. \quad & w_i - \theta_i h_i \geq 0; \\ & w_i - \theta_i h_i \geq w_j - \theta_i h_j. \end{aligned}$$

(c) Notice that the agent's payoff $w - \theta_i h$ has strictly decreasing differences in θ_i and h and strictly decreasing in θ_i, so the IR constraint for type H and the IC constraint for type L not to immitate type H are the binding constraints. Also, $h_L > h_H$. Therefore,

$$\begin{aligned} w_H &= \theta_H h_H; \\ w_L &= (\theta_H - \theta_L)h_H + \theta_L h_L. \end{aligned}$$

The employer chooses h_H and h_L to maximize

$$p_L(h_L/\theta_L - \theta_L h_L) + p_H[h_H/\theta_H - \theta_H h_H - (p_L/p_H)(\theta_H - \theta_L)h_H].$$

Clearly, $h_L^* = 1$. Also, when

$$\theta_H^{-1} - \theta_H - \frac{p_L}{p_H}(\theta_H - \theta_L) \geq 0, \tag{1}$$

$h_H^* = 1$, and when the LHS is negative, $h_H^* = 0$. When the LHS is zero, the employer is indifferent among all h_H.

(d) See (c) : the low type (here type H) will not be hired when the LHS of (1) is negative

Question 3

(a) First note the principal can choose y_H and y_L and set t to be arbitrarily low for any other realized output. Thus the principal's problem reduces to

$$\begin{aligned}
\max_{y,t}\ & p(y_H - t_H) + (1-p)(y_L - t_L) \\
\text{s.t. } & [\text{ICH}] \quad t_H - c(y_H - \sigma_H) \geq t_L - c(y_L - \sigma_H) \\
& [\text{ICL}] \quad t_L - c(y_L - \sigma_L) \geq t_H - c(y_H - \sigma_L) \\
& [\text{IRH}] \quad t_H - c(y_H - \sigma_H) \geq 0 \\
& [\text{IRL}] \quad t_L - c(y_L - \sigma_L) \geq 0
\end{aligned}$$

(b) We can first eliminate IRH:

$$t_H - c(y_H - \sigma_H) \geq t_L - c(y_L - \sigma_H) \geq t_L - c(y_L - \sigma_L) \geq 0$$

where the first inequality is ICH, the second follows from the fact that $\sigma_H > \sigma_L$ and $c(\cdot)$ is increasing, and the third inequality is IRL.

Next, it must be the case the IRL binds, otherwise the principal could lower t_H and t_L by equal amounts to earn profit without affecting the other constraints.

Third, ICH must bind. Otherwise it would be possible to lower t_H, which would improve the principal's profit while slackening ICL and not affecting IRL (which in turn, implies ICH is still satisfied).

Fourth, we can eliminate ICL by first establishing monotonicity. Adding the two IC constraints gives

$$\begin{aligned}
t_H + t_L - c(y_H - \sigma_H) - c(y_L - \sigma_L) &\geq t + H + t_L - c(y_H - \sigma_L) - c(y_L - \sigma_H) \\
\iff c(y_H - \sigma_L) + c(y_L - \sigma_H) &\geq c(y_H - \sigma_H) + c(y_L - \sigma_L) \implies y_H \geq y_L
\end{aligned}$$

where the conclusion follows from the fact that $c(\cdot)$ has a negative cross-partial w.r.t. y and σ. Using the fact that ICH binds,

$$\begin{aligned}
& t_H - c(y_H - \sigma_H) = t_L - c(y_L - \sigma_H) \\
\implies & t_H - c(y_H - \sigma_L) \leq t_L - c(y_L - \sigma_L)
\end{aligned}$$

again due to the negative cross-partial.

(c) Same method as 4.c, alternatively note that for a type σ agent, the first best choice of output, $y^*(\sigma)$, is given by

$$y^*(\sigma) = \arg\max_{y}\ y - c(y - \sigma).$$

so $w(y) = y$ implements the first best choice of output (note that $y^*(\sigma)=e^*+\sigma$ where $c'(e^*)=1$ so that $w(\sigma) = w(y(\sigma)) = y(\sigma) = e^*+\sigma$)

Question 4

(a) Consider the scheme $p(x) = x + A$ where A is chosen so that all firms want to produce. Then each firm maximizes $x - c(x, \theta) = b(x) - c(x, \theta)$ and thus chooses the first-best output

$$x^{FB}(\theta) = \frac{1}{\theta}. \tag{8}$$

Since the government's objective function is the social surplus $b(x) - c(\theta, x)$, she will implement the first-best.

(b) The program is as follows:

$$\begin{aligned} \max_{t_\theta, x_\theta} \quad & p(x_1 - t_1) + (1-p)(x_2 - t_2); \\ s.t. \quad & t_\theta - c(x_\theta, \theta) \geq 0; \\ & t_\theta - c(x_\theta, \theta) \geq t_{\theta'} - c(x_{\theta'}, \theta). \end{aligned}$$

First consider the problem of choosing (t_1, t_2) given (x_1, x_2) (where $x_1 \geq x_2$). Notice that the firm's profit is decreasing in θ, so only the participation constraint of the highest cost type is binding, so $t_2 = c(2, x - 2)$. The IC constraint implies that $t_1 - t_2 \in [c(x_1, 1) - c(x_2, 1), c(x_1, 2) - c(x_2, 2)]$. As the government wants to minimize t_1, $t_1 = c(x_1, 1) - c(x_2, 1) + t_2$. In other words, the low-cost type is indifferent between choosing x_1 and choosing x_2. The government's payoff under (x_1, x_2) is

$$p(x_1 - c(x_1, 1) + c(x_2, 1) - c(x_2, 2)) + (1-p)(x_2 - c(x_2, 2)).$$

It is straight forward to show that the optimal choice is $x_1^* = 1 = x^{FB}(1)$, and $x_2^* = \frac{1-p}{2-p} < x^{FB}(2)$. Therefore, output by the high-cost type is downward distorted.

(c) As before, the binding IR constraint is the constraint for the highest-cost type. Therefore,

$$t(\theta) = c(x_\theta, \theta) + \int_\theta^2 c_2(x_{\tilde{\theta}}, \tilde{\theta}) d\tilde{\theta} = \frac{1}{2}\theta(2-\theta)^2 + \int_\theta^2 \frac{1}{2}(2-\tilde{\theta})^2 d\tilde{\theta} = \frac{1}{2}\theta(2-\theta)^2 + \frac{1}{6}(2-\theta)^3.$$

Therefore, the payment scheme is

$$p(x) = t(2-x) = x^2 - \frac{1}{3}x^3.$$

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Spring 2017