Supermodularity

14. 126 Game Theory

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Based on Lectures by Paul Milgrom

Road Map

- Definitions: lattices, set orders, supermodularity…
- Optimization problems
- Games with Strategic Complements
  - Dominance and equilibrium
  - Comparative statics

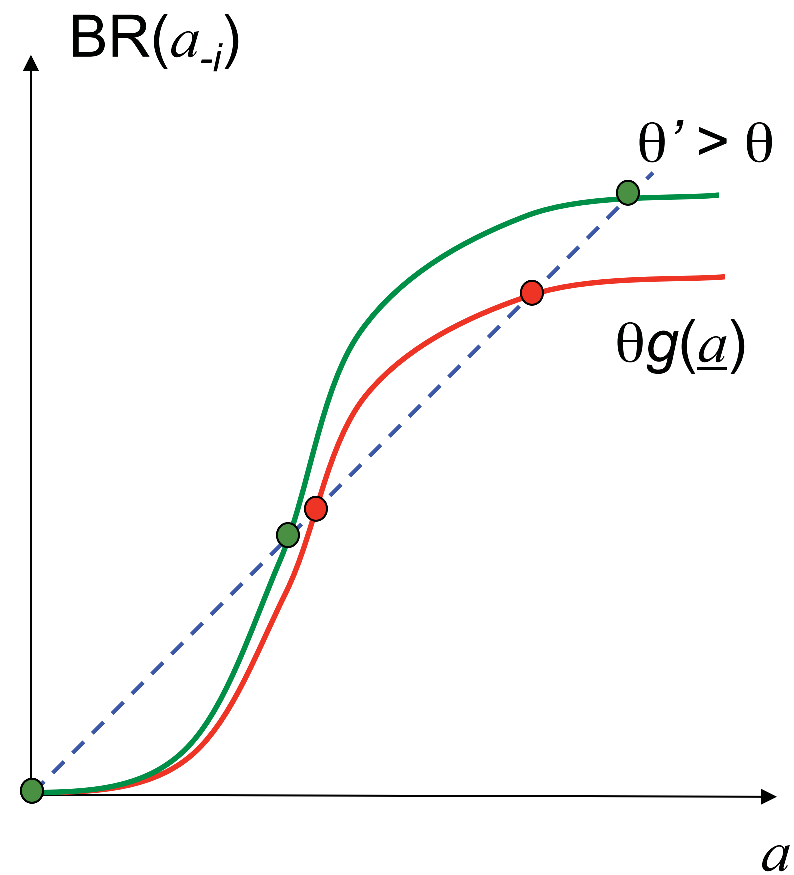
Two Aspects of Complements

- Constraints
  - Activities are complementary if doing one enables doing the other…
  - …or at least doesn't prevent doing the other.
    - This condition is described by sets that are sublattices.
- Payoffs
  - Activities are complementary if doing one makes it weakly more profitable to do the other…
    - This is described by supermodular payoffs.
  - …or at least doesn't change the other from being profitable to being unprofitable
    - This is described by payoffs satisfying a single crossing condition.

Example –Diamond search model

- A continuum of players
- Each i puts effort a_i, costing $a_i^2/2$;
- Pr i finds a match = $a_i g(\underline{a})$,
  - $\underline{a}$ is average effort of others
- The payoff from match is θ.

  $$U_i(a) = \theta a_i g(\underline{a}) - a_i^2/2$$
- Strategic complementarity: $BR(a_{-i}) = \theta g(\underline{a})$



Lattices

- Given a partially ordered set $(X,\geq)$, define
  - The *join* $x \vee y = \inf\{z \in X \mid z \geq x, z \geq y\}$.
  - The *meet* $x \wedge y = \sup\{z \in X \mid z \leq x, z \leq y\}$.
- $(X,\geq)$ is a *lattice* if it is closed under meet and join:

$$(\forall x, y \in X)\, x \wedge y \in X, x \vee y \in X$$

- Example: $X = \mathbf{R}^N$

$$x \geq y \text{ if } x_i \geq y_i, i = 1,\dots,N$$

$$(x \wedge y)_i = \min(x_i, y_i); i = 1,\dots,N$$

$$(x \vee y)_i = \max(x_i, y_i); i = 1,\dots,N$$

- $X=2^S$ with order given by inclusion; join=union, meet=intersection

Supermodularity

- (X,≥) is a ***complete lattice*** if for every non-empty subset S, a greatest lower bound inf(S) and a least upper bound sup(S) exist in X.

- A function f : X→**R** is ***supermodular*** if

$$(\forall x, y \in X) f(x) + f(y) \leq f(x \wedge y) + f(x \vee y)$$

- A function f is ***submodular*** if $-f$ is supermodular.
- If X = **R**, then any f is supermodular.

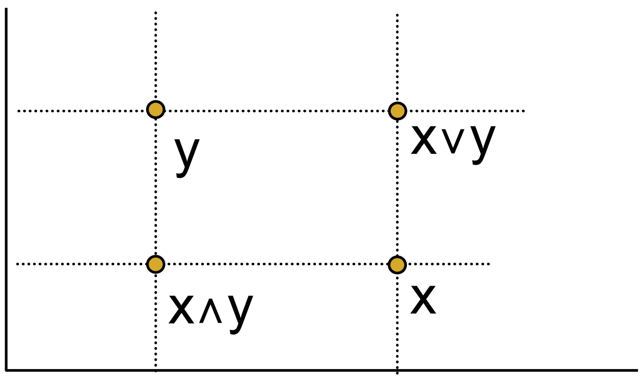
Complementarity

- Complementarity/supermodularity has equivalent characterizations:
  - Higher marginal returns

    $$f(x \vee y) - f(x) \geq f(y) - f(x \wedge y)$$

  - For smooth objectives, non-negative mixed second derivatives:

    $$\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0 \text{ for } i \neq j$$



Set order

- Given two subsets $S,T \subset X$, S is *as high as* T, written $S \geq T$, means

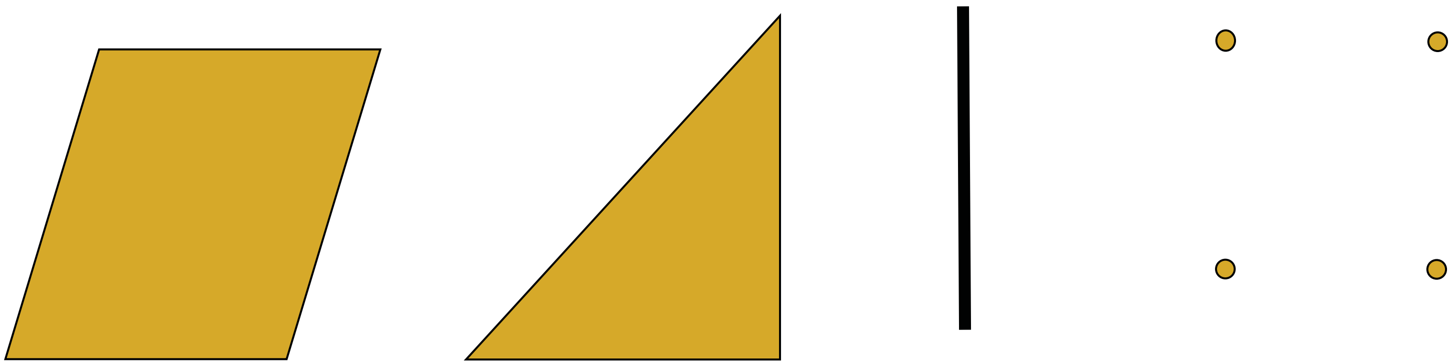
$$[x \in S \ \&\ y \in T] \Rightarrow [x \vee y \in S \ \ \&\ \ x \wedge y \in T]$$

- A function x* is *isotone* (or *weakly increasing*) if

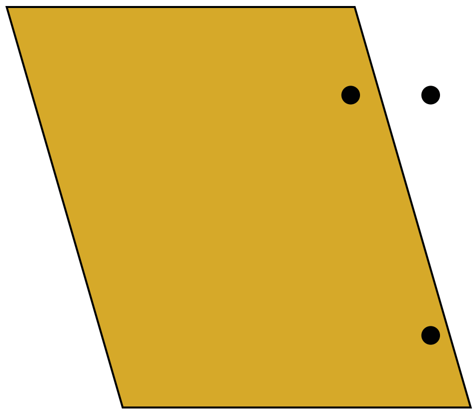
$$t \geq t' \Rightarrow x^*(t) \geq x^*(t')$$

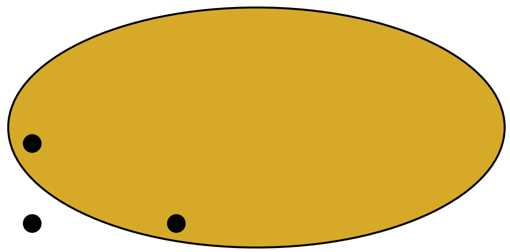
- A set S is a *sublattice* if $S \geq S$.

Sublattices of $\mathbf{R}^2$



Not Sublattices





Increasing differences

- Let $f:\mathbf{R}^N \rightarrow \mathbf{R}$. f is pairwise supermodular (or has increasing differences) iff
  - for all $n \neq m$ and x_{-nm}, the restriction $f(.,.,x_{-nm}):\mathbf{R}^2 \rightarrow \mathbf{R}$ is supermodular.
- Lemma: If f has increasing differences and $x_j \geq y_j$ for each j, then

$$f(x_i,x_{-i}) - f(y_i,x_{-i}) \geq f(x_i,y_{-i}) - f(y_i,y_{-i}).$$

- Proof:

$$\begin{aligned}
& f(x_1,x_{-1}) - f(x_1,y_{-1}) \\
&= \sum_{j>1} f(x_1,x_2,\ldots,x_j,y_{j+1},\ldots,y_n) - f(x_1,x_2,\ldots,x_{j-1},y_j,\ldots,y_n) \\
&\geq \sum_{j>1} f(y_1,x_2,\ldots,x_j,y_{j+1},\ldots,y_n) - f(y_1,x_2,\ldots,x_{j-1},y_j,\ldots,y_n) \\
&= f(y_1,x_{-1}) - f(y_1,y_{-1})
\end{aligned}$$

Pairwise Supermodular = Supermodular

- Theorem (Topkis). Let $f : \mathbf{R}^N \to \mathbf{R}$. Then, f is supermodular if and only if f is pairwise supermodular.
- Proof:
- $\Rightarrow$ by definition.
- $\Leftarrow$ Given x,y,

$$
\begin{aligned}
& f(x \vee y) - f(y) \\
&= \sum_i f(x_1 \vee y_1, \ldots, x_i \vee y_i, y_{i+1}, \ldots, y_n) - f(x_1 \vee y_1, \ldots, x_{i-1} \vee y_{i-1}, y_i, \ldots, y_n) \\
&= \sum_i f(x_1 \vee y_1, \ldots, x_{i-1} \vee y_{i-1}, x_i, y_{i+1}, \ldots, y_n) - f(x_1 \vee y_1, \ldots, x_{i-1} \vee y_{i-1}, x_i \wedge y_i, y_{i+1}, \ldots, y_n) \\
&\geq \sum_i f(x_1, \ldots, x_{i-1}, x_i, x_{i+1} \wedge y_{i+1}, \ldots, x_n \wedge y_n) - f(x_1, \ldots, x_{i-1}, x_i \wedge y_i, x_{i+1} \wedge y_{i+1}, \ldots, x_n \wedge y_n) \\
&= f(x) - f(x \wedge y)
\end{aligned}
$$

Supermodularity in product spaces

- Let $X = X_1 \times X_2 \times \ldots \times X_n$, $f : X \to \mathrm{R}$.
- Then, f is supermodular iff
  - For each i, the restriction of f to X_i is supermodular
  - f has increasing differences.

Monotonicity Theorem

- Theorem (Topkis). Let $f : X \times \mathbf{R} \rightarrow \mathbf{R}$ be a supermodular function and define
$$x^*(t) \equiv \underset{x \in S(t)}{\operatorname{argmax}} f(x,t).$$
If $t \geq t'$ and $S(t) \geq S(t')$, then $x^*(t) \geq x^*(t')$.
- Corollary. Let $f : X \times \mathbf{R} \rightarrow \mathbf{R}$ be a supermodular function and suppose $S(t)$ is a sublattice. Then, $x^*(t)$ is a sublattice.
- Proof of Corollary. Trivially, $t \geq t$, so $S(t) \geq S(t)$ and $x^*(t) \geq x^*(t)$.

Proof of Monotonicity Theorem

- [$t \geq t'$, $S(t) \geq S(t') \Rightarrow x^*(t) \geq x^*(t')$, where $x^*(t)$=argmax$_{x \in S(t)}$f(x,t)]
- Suppose that f is supermodular

  and that $x \in x^*(t), x' \in x^*(t'), t \geq t'$.
- Then, $(x \wedge x') \in S(t'), (x \vee x') \in S(t)$

  So, $f(x,t) \geq f(x \vee x', t)$ and $f(x',t') \geq f(x \wedge x', t')$.

- If either inequality is strict then their sum contradicts supermodularity:

$$f(x,t) + f(x',t') > f(x \wedge x', t') + f(x \vee x', t).$$

Application: Pricing Decisions

- A monopolist facing demand $D(p,t)$ produces at unit cost c.

$$p^*(c,t) = \text{argmax}_p\ (p - c)D(p,t)$$
$$= \text{argmax}_p\ \log(p - c) + \log(D(p,t))$$

- $p^*(c,t)$ is always isotone in c.
- $p^*(c,t)$ is isotone in t if $\log(D(p,t))$ is supermodular in (p,t),
  - i.e. supermodular in $(\log(p),t)$,
  - i.e. increases in t make demand less elastic:

$$\frac{\partial \log D(p,t)}{\partial \log(p)} \text{ nondecreasing in } t$$

Application: Auction Theory

- A firm's value of winning an item at price *p* is $U(p,t)$, where *t* is the firm's type. (Losing is normalized to zero.) A bid of *p* wins with probability $F(p)$.
- Question: Can we conclude that $p(t)$ is nondecreasing, without knowing *F*?

$$p_F^*(t) = \underset{p}{\operatorname{argmax}}\, U(p,t)F(p)$$

$$= \underset{p}{\operatorname{argmax}} \log\big(U(p,t)\big) + \log\big(F(p)\big)$$

- Answer: Yes, if $\log(U(p,t))$ is supermodular.

Convergence in Lattices

- Consider a complete lattice $(X,\geq)$.
- Want to define continuity for $f : X \to R$. Consider a topology on X in which
  - For any sequence $(x_m)_{m>0}$ with $x_m \geq x_{m+1}$ $\forall$m,

    $$x_m \to \inf \{ x_m : m > 0\} = \lim x_m$$

  - For any sequence $(x_m)_{m>0}$ with $x_{m+1} \geq x_m$ $\forall$m,

    $$x_m \to \sup \{ x_m : m > 0\} = \lim x_m$$

- f is continuous if for every monotone (x_m),

  $$f(\lim x_m) = \lim f(x_m).$$

Supermodular Games

Formulation

A supermodular game (N,X,u)

- N players (infinite is okay)
- Strategy sets $(X_n, \geq_n)$ are complete lattices
  - $\underline{x}_n = \min X_n, \overline{x}_n = \max X_n$
- Payoff functions $U_n(x)$ are
  - continuous
  - supermodular in own strategy and has increasing differences with others' strategies

$$(\forall n)(\forall x_n, x'_n \in X_n)(\forall x_{-n} \geq x'_{-n} \in X_{-n})$$

$$U_n(x) + U_n(x') \leq U_n(x \wedge x') + U_n(x \vee x')$$

Differentiated Bertrand Oligopoly

- Linear/supermodular oligopoly

Demand: $Q_n(x) = A - ax_n + \sum_{j \neq n} b_j x_j$

Profit: $U_n(x) = \left(x_n - c_n\right) Q_n(x)$

$$\frac{\partial U_n}{\partial x_m} = b_m(x_n - c_n) \text{ which is increasing in } x_n$$

Linear Cournot Duopoly

- Inverse demand: $P(x) = A - x_1 - x_2$

  $U_n(x) = x_n P(x) - C_n(x_n)$

  $$\frac{\partial U_n}{\partial x_m} = -x_n$$

- Linear Cournot duopoly (but not more general oligopoly) is supermodular if one player's strategy set is given the reverse of its usual order.

Analysis of Supermodular Games

- Extremal best response functions

$$B_n(x) = \max\left(\operatorname*{argmax}_{x_n' \in X_n} U_n(x_n', x_{-n})\right)$$

$$b_n(x) = \min\left(\operatorname*{argmax}_{x_n' \in X_n} U_n(x_n', x_{-n})\right)$$

  - By Topkis's Theorem, these are isotone functions.
- Lemma:

$$\neg\left[x_n \geq b_n(\underline{x})\right] \Rightarrow \left[x_n \text{ is strictly dominated by } b_n(\underline{x}) \vee x_n\right]$$

- Proof.

If $\neg\left[x_n \geq b_n(\underline{x})\right]$, then

$$U_n(x_n \vee b_n(\underline{x}), x_{-n}) - U_n(x_n, x_{-n}) \geq U_n(b_n(\underline{x}), \underline{x}_{-n}) - U_n(x_n \wedge b_n(\underline{x}), \underline{x}_{-n}) > 0$$

Supermodularity + increasing differences

Rationalizability & Equilibrium

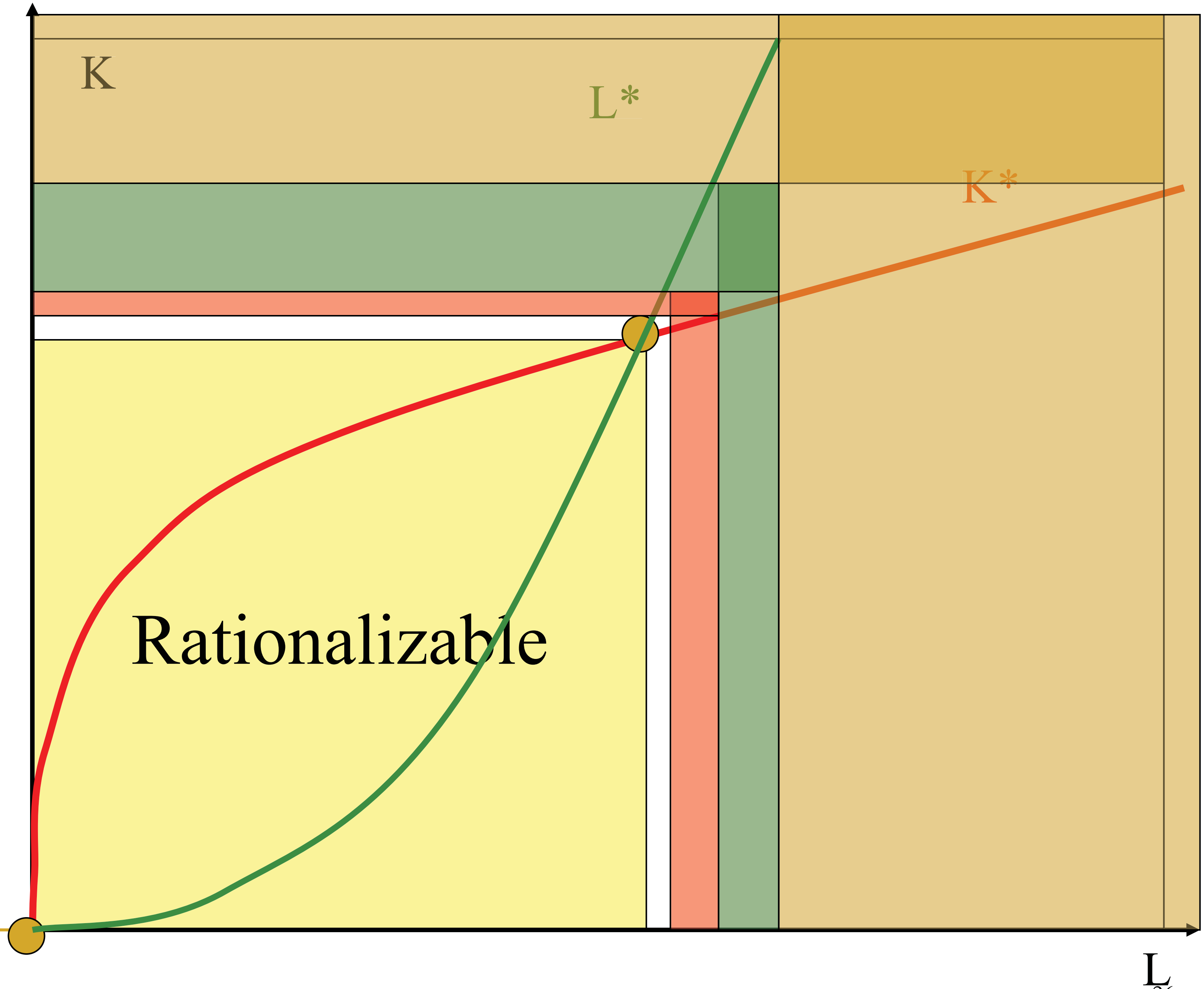
- Theorem (Milgrom & Roberts): The smallest rationalizable strategies for the players are given by
$$\underline{z} = \lim_{k\to\infty} b^k(\underline{x})$$
Similarly the largest rationalizable strategies for the players are given by
$$\bar{z} = \lim_{k\to\infty} B^k(\bar{x})$$
Both are Nash equilibrium profiles.

- Corollary: there exist pure strategy Nash equilibria $\bar{z}$ and $\underline{z}$ s.t.
  - For each rationalizable x, $\bar{z} \geq x \geq \underline{z}$.
  - For each Nash equilibrium x, $\bar{z} \geq x \geq \underline{z}$.

Partnership Game

- Two players; employer (E) and worker (W)
- E and W provide K and L, resp.
- Output: $f(K,L) = K^{\alpha}L^{\beta}$, $0 < \alpha,\beta,\alpha+\beta < 1$.
- Payoffs of E and W:

$$f(K,L)/2 - K, \; f(K,L)/2 - L.$$


K
L*
K*
Rationalizable
L

Proof

- $b^k(\underline{x})$ is isotone and X is complete, so limit $\underline{z}$ of $b^k(\underline{x})$ exists.
- By continuity of payoffs, its limit is a fixed point of b, and hence a Nash equilibrium.
- $x_n \ngeq \underline{z}_n \Rightarrow \mathrm{x_n} \ngeq b_n^k(\underline{x})$ for some k, and hence x_n is deleted during iterated deletion of dominated strategies.

Comparative Statics

- Theorem. (Milgrom & Roberts) Consider a family of supermodular games with payoffs parameterized by t. Suppose that for all n, x_{-n}, $U_n(x_n,x_{-n};t)$ is supermodular in (x_n,t). Then

$$\bar{z}(t), \underline{z}(t) \text{ are isotone.}$$

- Proof. By Topkis's theorem, $b_t(x)$ is isotone in t. Hence, if $t > t'$,

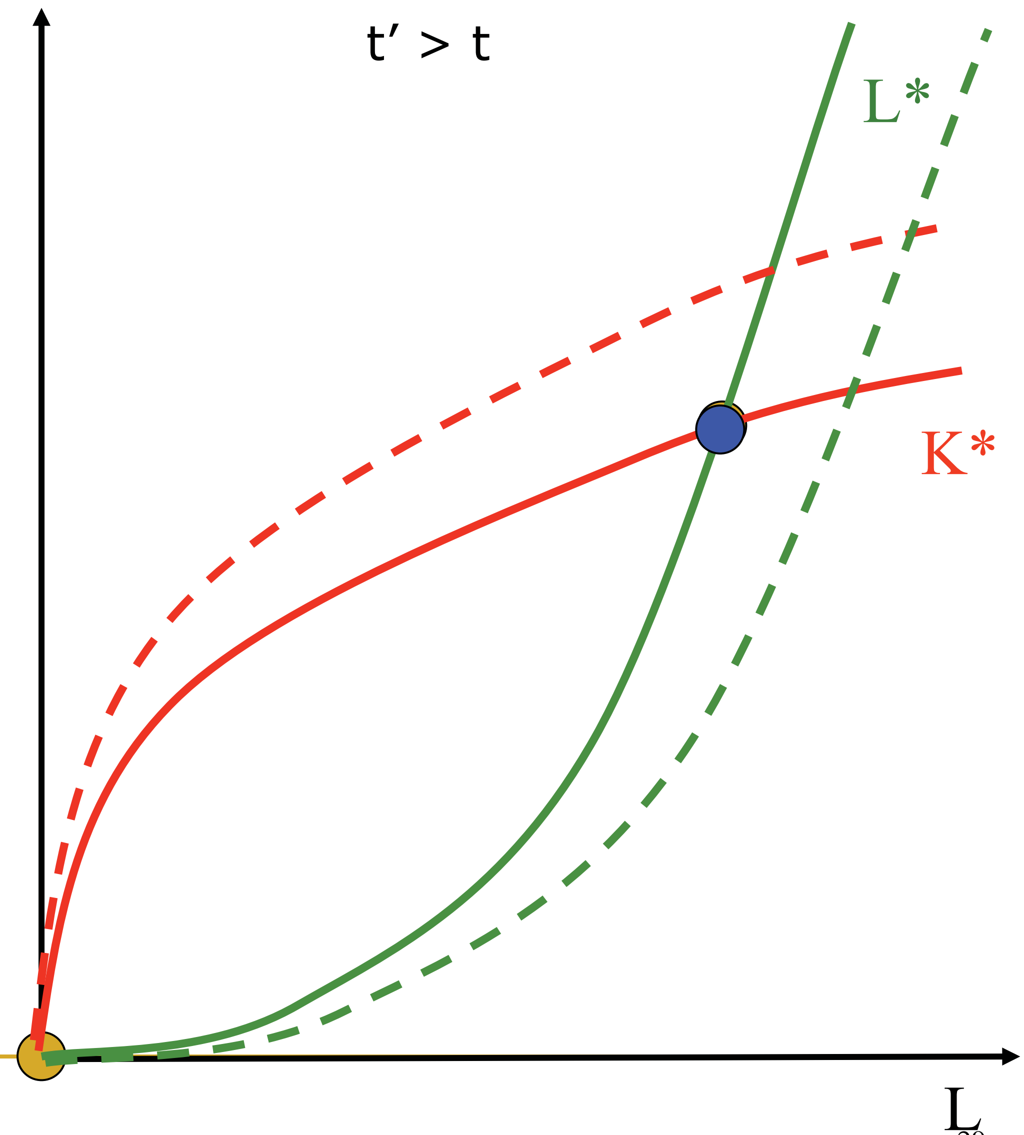
$$b_t^k(\underline{x}) \geq b_{t'}^k(\underline{x})$$

$$\underline{z}(t) = \lim_{k\to\infty} b_t^k(\underline{x}) \geq \lim_{k\to\infty} b_{t'}^k(\underline{x}) \geq \underline{z}(t')$$

and similarly for $\bar{z}$.

Example – partnership game

- $f(K,L) = tK^{\alpha}L^{\beta}$,



Monotone supermodular games

- $G = (N,T,A,u,p)$
- $T = T_0 \times T_1 \times \ldots \times T_n$ ($\subseteq R^M$)
- A_i compact sublattice of R^K
- $u_i : A \times T \rightarrow R$
  - $u_i(a,.): T \rightarrow R$ is measurable
  - $u_i(.\ ,t): A \rightarrow R$ is continuous, bounded, supermodular in a_i, has increasing differences in a and in (a_i,t)
- $p(.|t_i)$ is increasing function of t_i—in the sense of 1st-order stochastic dominance (e.g. p is affiliated).
- Theorem: There exist BNE s^* and s^{**} such that
  - For each BNE s, $s^* \geq s \geq s^{**}$.
  - Both s^* and s^{**} are isotone.

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