Economics of Networks
Introduction to Game Theory: Part 2

Evan Sadler

Massachusetts Institute of Technology

Agenda

Recap of last time

Mixed strategies and mixed strategy equilibrium

Existence of Nash Equilibria

Extensive form games and subgame perfection

Reading: Osborne chapters 4-6

Recap

Rational choice:

- Agents described by preferences, can represent as utility function
- With uncertainty, maximize expected utility

Dominant and dominated strategies

- Intuitive game solutions
- Can't always get a unique prediction

Pure strategy Nash Equilibrium

- Everyone plays a best response
- Doesn't always exist...

Nonexistence

Recall the matching pennies game:

	Heads	Tails
Heads	$(-1, 1)$	$(1, -1)$
Tails	(1,-1)	$(-1, 1)$

No pure strategy Nash Equilibrium

How would you play?

Nonexistence

Alternative interpretation: Penalty Kicker and Goalie

Kicker / Goalie	Left	Right
Left	$(-1, 1)$	$(1, -1)$
Right	(1,-1)	$(-1, 1)$

Is it a good strategy for the kicker to always kick to the left side of the net?

Empirical evidence suggests that most penalty kickers "randomize"

- Mixed strategies

Mixed Strategies

Let Σ_i denote the set of all lotteries over pure strategies in S_i

- In our example, a mixed strategy is a probability of kicking (or diving) left

Write $\sigma_i \in \Sigma_i$ for the strategy of i

Write $\sigma \in \Sigma = \prod_{i \in N} \Sigma_i$ for a strategy profile

- Implicitly assume players randomize independently
- $\sigma_{-i} \in \Sigma_{-i}$ denotes strategies of other players

Payoff is expected utility:

$$u_i(\sigma) = \int_S u_i(s) \mathrm{d}\sigma(s)$$

Mixed Strategy Nash Equilibrium

Definition

A mixed strategy profile σ^* is a Nash Equilibrium if for each player i and all $\sigma_i \in \Sigma_i$

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

The strategy σ_i^* is a best response to σ_{-i}^*

- Best response to correct conjecture

Space of lotteries is large, how do we tell we have an equilibrium?

Mixed Strategy Nash Equilibrium

Proposition

In a normal form game, the profile $\sigma^* \in \Sigma$ is a Nash Equilibrium if and only if for each player i, every pure strategy in the support of σ_i^* is a best response to σ_{-i}^*.

We only need to check pure strategy deviations

Proof idea: If we put positive probability on a strategy that is not a best response, shifting that probability to a best response strictly increases utility.

Mixed Strategy Nash Equilibrium

Consequence: every action in support of i's equilibrium mixed strategy yields same expected payoff

Extends to infinite games

Matching pennies: unique mixed Nash equilibrium, players put probability $\frac{1}{2}$ on heads

	Heads	Tails
Heads	$(-1,1)$	$(1,-1)$
Tails	(1,-1)	$(-1,1)$

Example: Work or Shirk

Recall the partnership game:

	Work	Shirk
Work	$(2, 2)$	$(-1, 1)$
Shirk	(1,-1)	$(0, 0)$

Two pure strategy equilibria

- Are there mixed equilibria?

Yes! Both randomize with probability $\frac{1}{2}$

- Expected payoff of $\frac{1}{2}$

Interpretation of Mixed Equilibria

Deliberate choice to randomize

- Recall our penalty kicker
- Bluffing in poker

Concern: indifference between strategies in support

- Continuum of best responses

Steady state of a learning process

Distribution of outcomes in a perturbed game with pure strategy best responses

- "Purification"

Nash's Theorem

Theorem

Every finite game has a mixed strategy Nash equilibrium.

Implication: games like matching pennies always have mixed equilibria

Why do we care?

- Without existence, studying properties of equilibria is difficult (maybe meaningless)
- Knowing existence, we can just try to find the equilibria

Tools: Weierstrass's Theorem

Theorem (Weierstrass)

Let A be a nonempty compact subset of a finite dimensional Euclidean space, and let $f : A \to \mathbb{R}$ be a continuous function. The function f attains a maximum and a minimum in A.

Recall definition of compactness: every sequence has a convergent subsequence

- Continuity ensures sup and inf are contained in the image $f(A)$

Tools: Kakutani's Fixed Point Theorem

Theorem (Kakutani)

Let $f : A \rightrightarrows A$ be a correspondence, i.e. $x \in A \implies f(x) \subset A$, satisfying:

- *A is a non-empty compact and convex subset of a finite dimensional Euclidean space*
- *$f(x)$ is non-empty for all $x \in A$*
- *$f(x)$ is convex valued*
- *$f(x)$ has a closed graph, i.e. $(x_n, y_n) \to (x, y)$ with $y_n \in f(x_n)$ implies $y \in f(x)$*

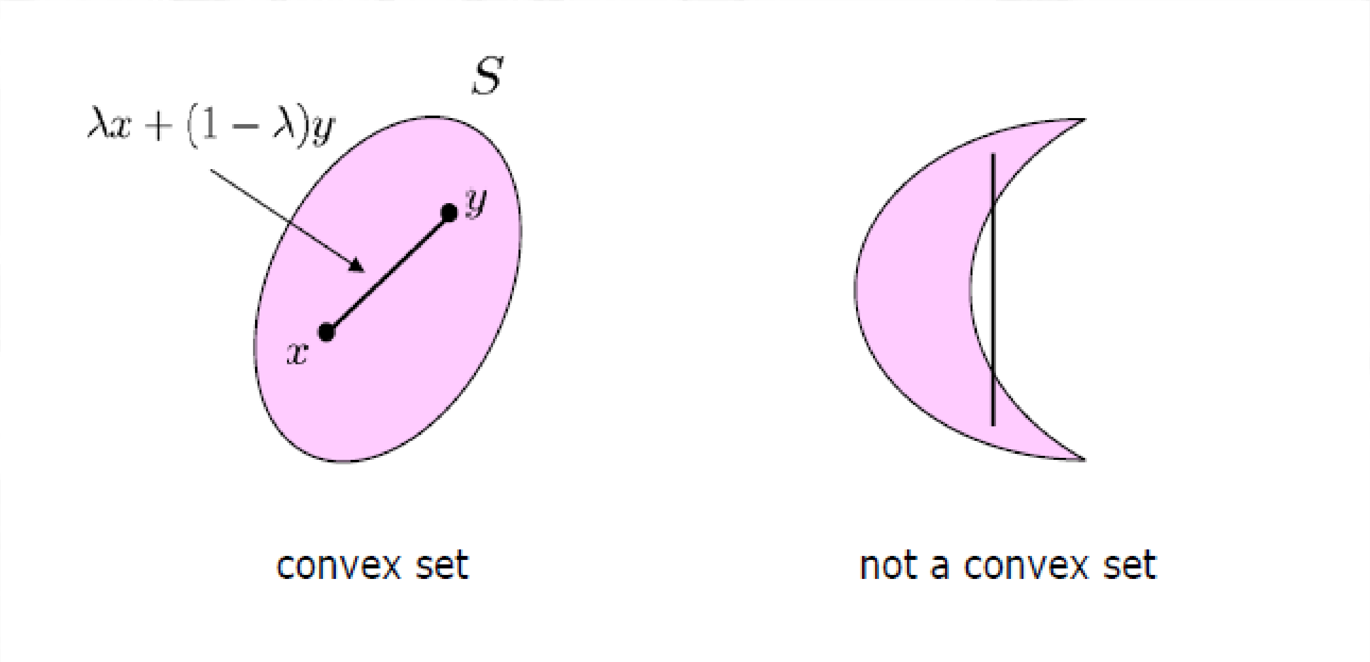
Then f has a fixed point: there exists $x \in A$ such that $x \in f(x)$

Definitions

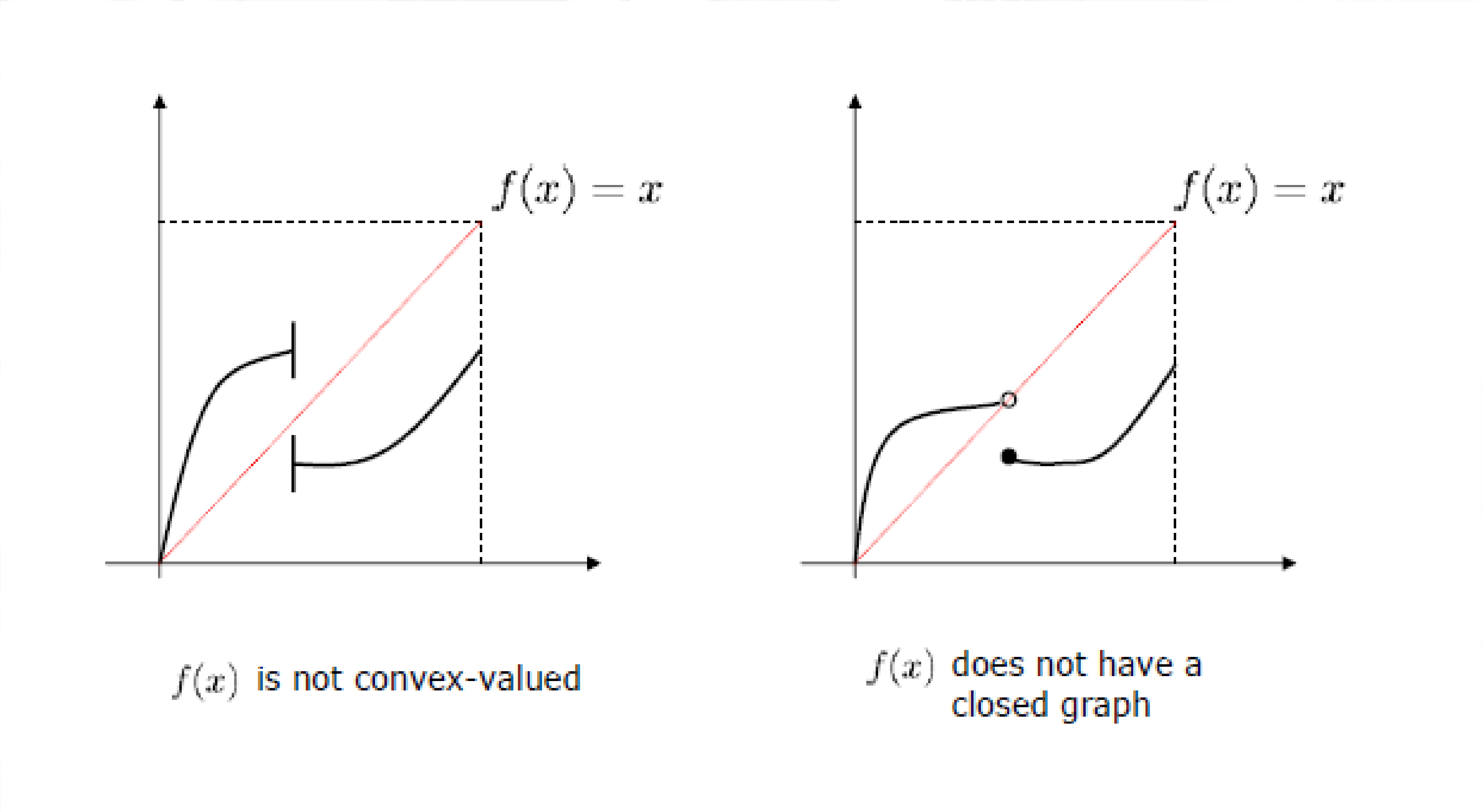
A set in Euclidean space is compact iff it is bounded and closed

- Every infinite sequence has a convergent subsequence

A set S is convex if for any $x, y \in S$ and any $\lambda \in [0, 1]$, we have $\lambda x + (1 - \lambda) y \in S$.



Kakutani's Fixed Point Theorem, Illustration



Proof of Nash's Theorem

Recall σ^* is a mixed strategy Nash Equilibrium if for every player i and every $\sigma_i \in \Sigma_i$,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*)$$

Define best response correspondence $B_i : \Sigma_{-i} \rightrightarrows \Sigma_i$ for player i:

$$B_i(\sigma_{-i}) = \{\sigma_i' \in \Sigma_i \,:\, u_i(\sigma_i', \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i}),\, \forall\, \hat{\sigma}_i \in \Sigma_i\}$$

Set of best response correspondences

$$B(\sigma) = \{B_i(\sigma_{-i})\}_{i \in N}$$

Proof, continued

We apply Kakutani's fixed point theorem to the correspondence $B : \Sigma \rightrightarrows \Sigma$

Need to show:

- Σ is compact, convex, and non-empty
- $B(\sigma)$ is non-empty
- $B(\sigma)$ is convex-valued
- $B(\sigma)$ has a closed graph

$\Sigma = \prod_{i \in N} \Sigma_i$ is compact, convex, and non-empty by definition

- Σ_i is a simplex of dimension $|S_i| - 1$

Proof, continued

$B(\sigma)$ is non-empty by Weierstrass's theorem

- Σ_i is non-empty and compact, so u_i attains its maximum for each i

$B(\sigma)$ is convex-valued, meaning $B_i(\sigma_{-i})$ is convex for each i

- Recall proposition on pure strategy deviations
- If σ_i' and σ_i'' both maximize u_i, any mixture does as well

For any $\hat{\sigma}_i$, we have

$$\begin{aligned} u_i\left(\lambda\sigma_i' + (1-\lambda)\sigma_i'', \sigma_{-i}\right) &= \lambda u_i(\sigma_i', \sigma_{-i}) + (1-\lambda)u_i(\sigma_i'', \sigma_{-i}) \\ &\geq \lambda u_i(\hat{\sigma}_i, \sigma_{-i}) + (1-\lambda)u_i(\hat{\sigma}_i, \sigma_{-i}) \\ &= u_i(\hat{\sigma}_i, \sigma_{-i}) \end{aligned}$$

Proof, continued

$B(\sigma)$ has a closed graph

- Suppose not
- Then, there exists $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$ with $\hat{\sigma}^n \in B(\sigma^n)$, but $\hat{\sigma} \notin B(\sigma)$
- That is, there exists i such that $\hat{\sigma}_i \notin B_i(\sigma_{-i})$
- Since $\hat{\sigma} \notin B_i(\sigma_{-i})$, there exists $\sigma_i' \in \Sigma_i$ and $\epsilon > 0$ such that

$$u_i(\sigma_i', \sigma_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 3\epsilon$$

- By continuity, for sufficiently large n we have

$$u_i(\sigma_i', \sigma_{-i}^n) \geq u_i(\sigma_i', \sigma_{-i}) - \epsilon$$

Proof, continued

Combining the last two inequalities, we have

$$u_i(\sigma'_i, \sigma^n_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 2\epsilon \geq u_i(\hat{\sigma}^n_i, \sigma^n_{-i}) + \epsilon$$

This contradicts assumption that $\hat{\sigma}^n_i \in B_i(\sigma^n_{-i})$

- We conclude that B has a closed graph

By definition, σ^* is a mixed strategy equilibrium if $\sigma^* \in B(\sigma^*)$

Equilibrium existence follows from Kakutani's theorem

Equilibrium Existence in Infinite Games

A similar theorem gives existence of pure strategy equilibria in infinite games

Theorem (Debreu, Glicksburg, Fan)

Consider an infinite normal form game $(N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ such that for each $i \in N$:

- *S_i is compact and convex*
- *$u_i(s_i, s_{-i})$ is continuous in s_{-i}*
- *$u_i(s_i, s_{-i})$ is continuous and concave in s_i*

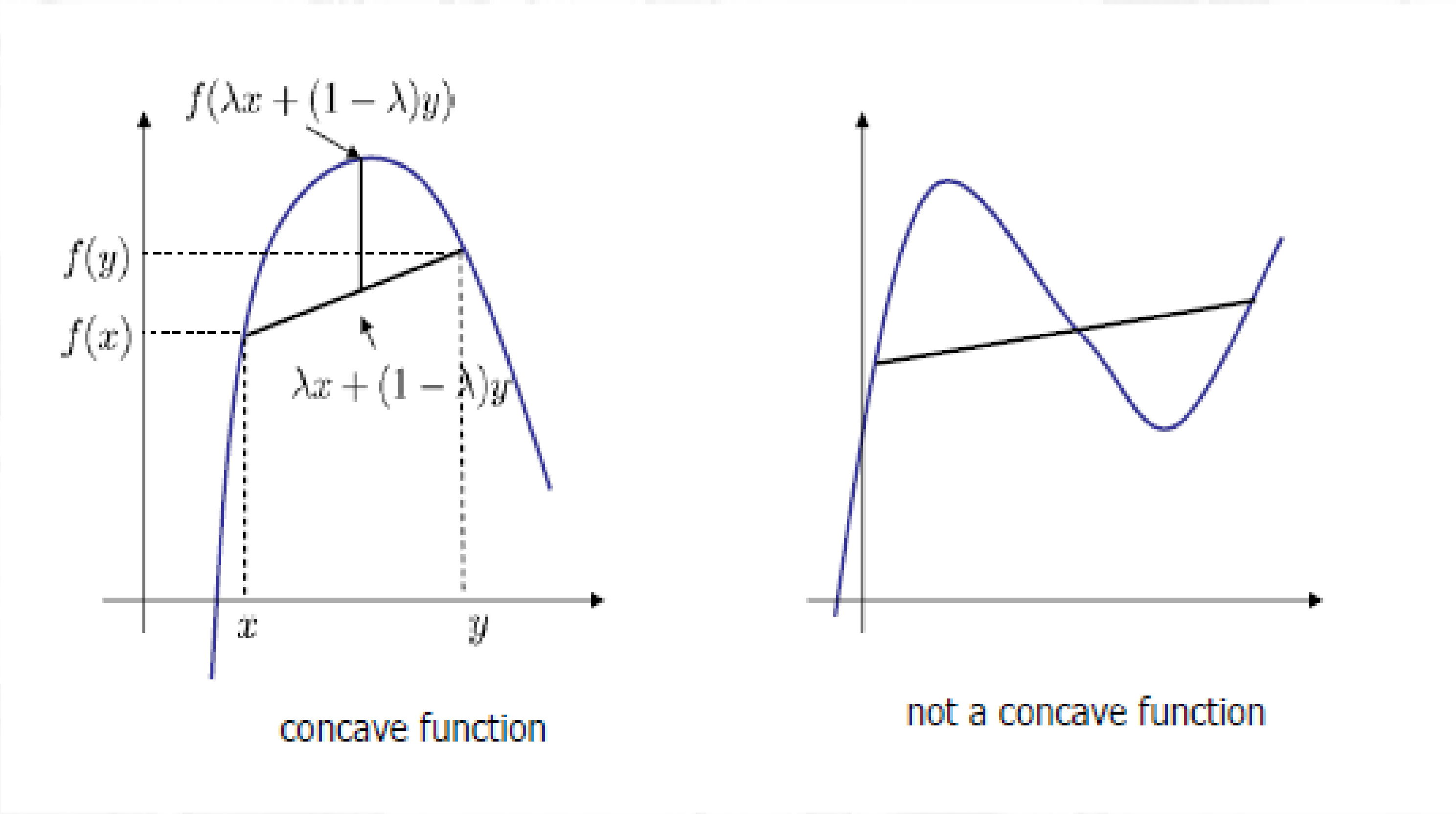
Then a pure strategy Nash Equilibrium exists.

Proof left as exercise

Definitions

Suppose S is a convex set. Then a function $f : S \to \mathbb{R}$ is concave if for any $x, y \in S$ and $\lambda \in [0, 1]$ we have

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$$



More Existence Questions

Can we relax concavity?

Example:

- Two players simultaneously pick locations $s_1, s_2 \in \mathbb{R}^2$ on the unit circle
- Player 1's payoff strictly increasing function of distance between players
- Player 2's payoff strictly decreasing function of distance between players

No pure strategy Nash Equilibrium

- Can express strategies as a compact convex set, but payoffs are not concave

There are mixed strategy equilibria...

A Stronger Theorem

Theorem (Glicksberg)

Consider an infinite normal form game such that

- S_i *is compact and convex for each* i
- $u_i(s_i, s_{-i})$ *is continuous in both arguments*

Then a mixed strategy Nash Equilibrium exists.

Proof is beyond scope of this class

Extensive Form Games

Up to now, we have ignored dynamics

Extensive form games capture strategic situations with multiple actions in sequence

- For now, focus on games with observable actions

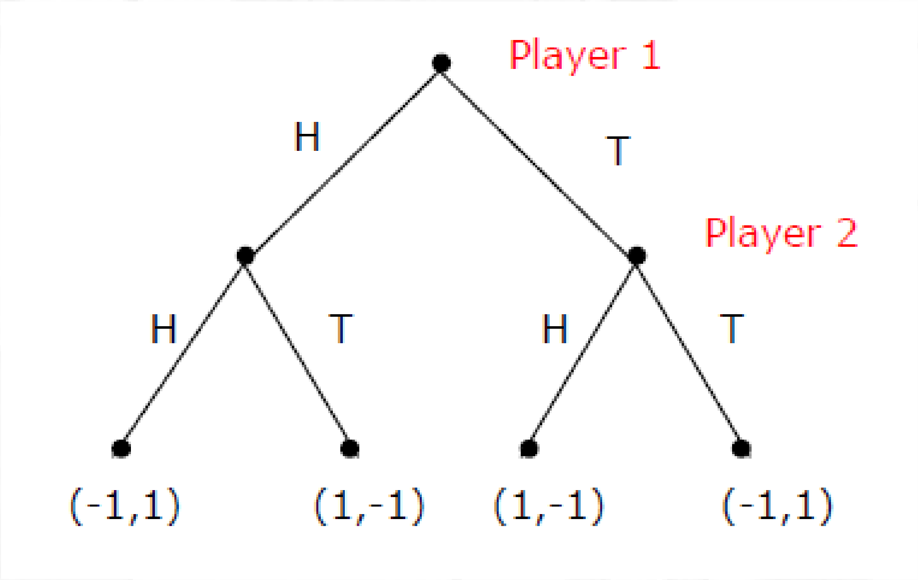
Represent extensive form using a game tree

- Keep track of possible histories

Definitions

Extensive form game is a collection $(N, H, Z, \{A_i^h\}_{\substack{i \in N \\ h \in H}}, \{u_i\}_{i \in N})$

- Set of players N
- Set of non-terminal histories H
- Set of terminal histories Z
- Actions A_i^h for each player i at each non-terminal history h
- Payoff function u_i giving payoff to i at each terminal history



Strategies in Extensive Form Games

A strategy for player i is a map s_i giving an action for each non-terminal history $h \in H$

- Strategy is a complete contingent plan

In example, player 1 has two strategies, H and T

How many does player 2 have?

- Four: HH, HT, TH, TT

Strategies in Extensive Form Games

Can use strategies to express extensive form game in normal form

- Action in normal form game is choice of a complete contingent plan

Normal form of two-stage matching pennies:

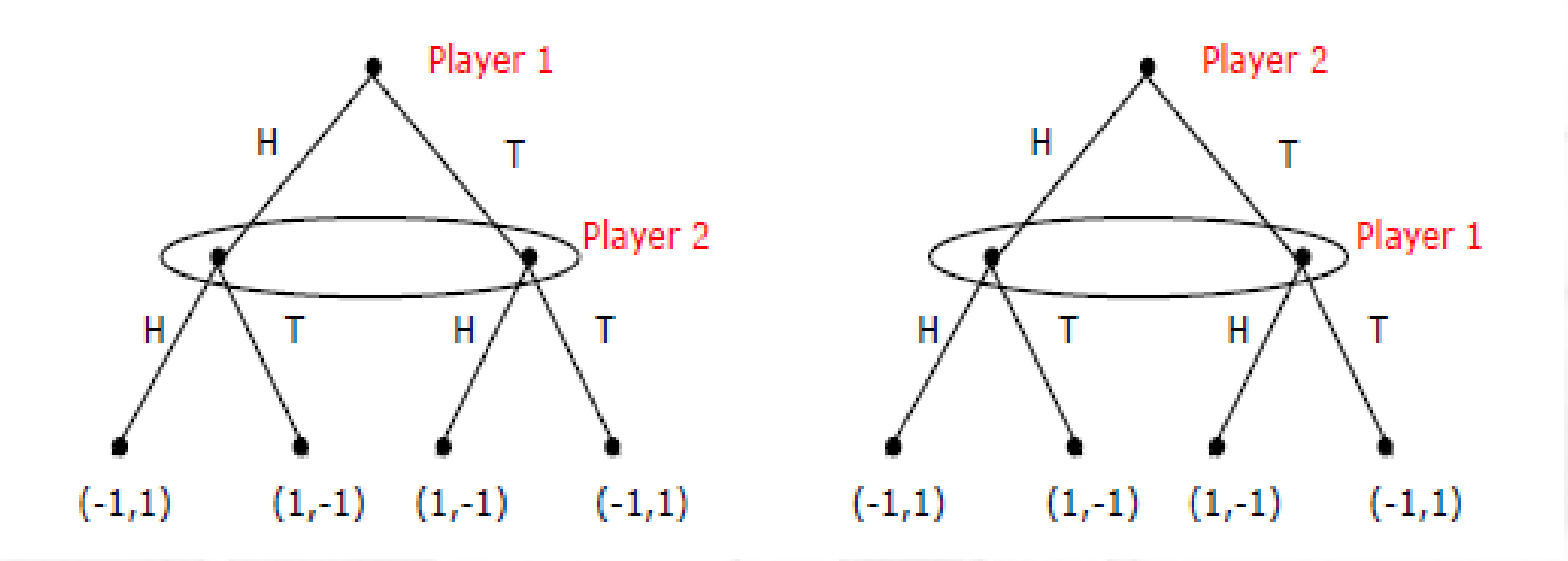
Player 1 / Player 2	HH	HT	TH	TT
Heads	$(-1,1)$	$(-1,1)$	$(1,-1)$	$(1,-1)$
Tails	$(1,-1)$	$(-1,1)$	$(1,-1)$	$(-1,1)$

Sidebar: Normal Form to Extensive Form

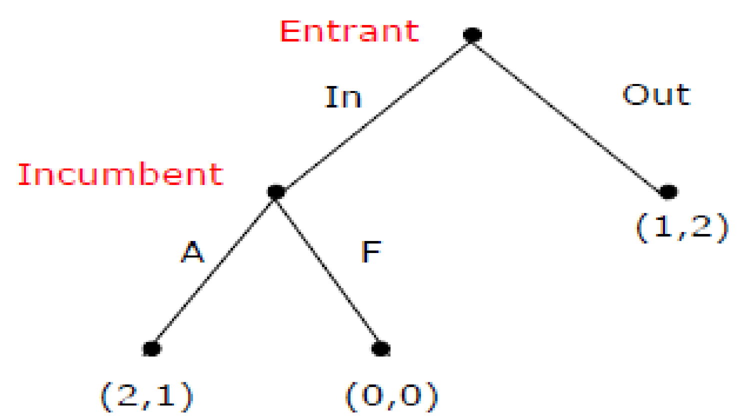
Recall the original matching pennies example: players choose heads/tails simultaneously

Can represent using a game tree by adding information sets

- Player cannot distinguish two decision nodes in same information set



Example: Entry Deterrence



Normal form representation:

Entrant / Incumbent	Accommodate	Fight
In	$(2,1)$	$(0,0)$
Out	$(1,2)$	$(1,2)$

Two pure Nash equilibria: (In, A) and (Out, F).

Are Both Equilibria Reasonable?

Equilibrium (Out, F) sustained by noncredible threat

- After observing entry, best response is to accommodate

Refinement by "subgame perfection"

- Strategy must be optimal going forward from any history
- Solve game via backward induction

Need to formally define a subgame

Subgame Perfect Equilibrium

Recall an extensive form game is expressed as a game tree

- Let V_G denote the set of nodes

An information set $X \subseteq V_G$ is a successor of node y (written $X \succ y$) if we can reach X through y

Definition

A subgame G_x of G is the set of nodes $V_G^x \subset V_G$ that are successors of some node $x \in V_G^x$ and not of any $z \notin V_G^x$

Subgame Perfect Equilibrium

A restriction of a strategy profile σ to the subgame G_x, written $\sigma_{|G_x}$ is the profile implied by σ in the subgame G_x

Definition

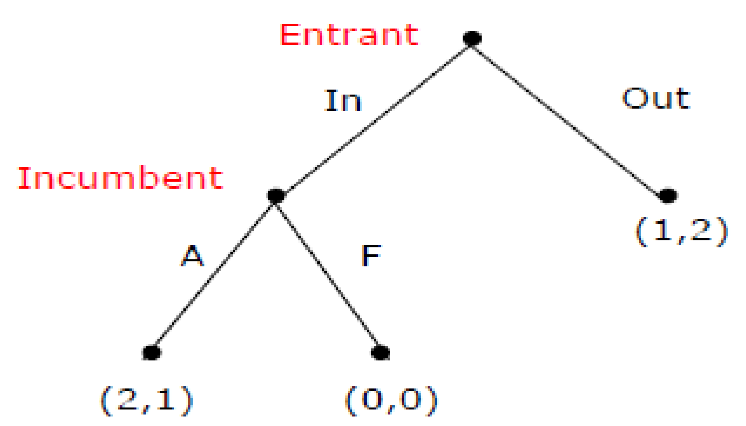
A strategy profile σ^* is a subgame perfect Nash equilibrium of G if for any subgame G_x of G, $\sigma^*_{|G_x}$ is a Nash equilibrium of G_x.

Rules out non-credible threats

How to find subgame perfect equilibria?

Backward Induction

Backward induction: start from the last subgames, find Nash equilibria of those, then work backwards towards the beginning of the game



Existence of Subgame Perfect Equilibria

Theorem

Every finite perfect information extensive form game G has a pure strategy SPE

Note: perfect information means all information sets contain exactly one node

Theorem

Every finite extensive form game G has a SPE

Follow's from Nash's theorem

Value of Commitment

What if the incumbent firm could commit to fight?

Could adjust the game tree to allow this

- Now the unique SPE is (Out, F)
- Incumbent is better off

Consider a dynamic version of Cournot competition

- Firm 1 commits to a quantity of output first
- Only after this does firm 2 choose a quantity

Stackleberg Competition

Recall the two firms will face a market price $p = 1 - q_1 - q_2$

- Firm i earns $q_i(p - c)$

Backward induction: solve firm 2's problem

- First order condition implies $q_2 = \frac{1-q_1-c}{2}$ as before

Firm 1 chooses q_1 to maximize

$$q_1(p-c) = q_1\left(1 - q_1 - \frac{1-q_1-c}{2} - c\right) = q_1\left(\frac{1-q_1-c}{2}\right)$$

giving $q_1 = \frac{1-c}{2}$.

- Total output is higher in the Stackleberg equilibrium (why?)

Recap

Nash equilibrium will be our workhorse solution concept

- Can essentially always guarantee existence of a (mixed strategy) equilibrium

Will employ refinements, especially in dynamic games, where appropriate

Next time: a network application of basic game theory

- Traffic routing