6.207/14.15: Networks
Lectures 4, 5 & 6: Linear Dynamics, Markov Chains, Centralities

Outline

Dynamical systems. Linear and Non-linear.

Convergence. Linear algebra and Lyapunov functions.

Markov chains.

Positive linear systems. Perron-Frobenius.
Random walk on graph.

Centralities.

Eigen centrality. Katz centrality.
Page Rank. Hubs and Authorities.

Reading:

Newman, Chapter 6 (Sections 6.13-14).
Newman, Chapter 7 (Sections 7.1-7.5).

Dynamical systems

Discrete time system: time indexed by k

let $x(k) \in \mathbb{R}^n$ denote system state

e.g. amount of labor, steele and coal available in an economy

System dynamics: for any $k \geq 0$

$$x(k+1) = F(x(k)) \tag{1}$$

for some $F : \mathbb{R}^n \to \mathbb{R}^n$

Primary questions:

Is there an equilibrium $x^\star \in \mathbb{R}^n$, i.e. $x^\star = F(x^\star)$.

If so, does $x(k) \to x^\star$ and how quickly.

Linear dynamical systems

Linear system dynamics: for any $k \geq 0$

$$x(k+1) = Ax(k) + b \tag{2}$$

for some $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$
example: Leontif's input-output model of economy

We'll study

Existence and characterization of equilibrium.
Convergence.

Initially, we'll consider $b = \mathbf{0}$

Later, we shall consider generic $b \in \mathbb{R}^n$

Linear dynamical systems

Consider

$$\begin{aligned} x(k) &= Ax(k-1) \\ &= A \times Ax(k-2) \\ &\cdots \\ &= A^k x(0) \end{aligned}$$

So what is A^k?

For $n = 1$, let $A = a \in \mathbb{R}_+$:

$$x(k) = a^k x(0) \overset{k \to \infty}{\rightarrow} \begin{cases} 0 \text{ if } 0 \leq a < 1 \\ x(0) \text{ if } a = 1 \\ \infty \text{ if } 1 < a. \end{cases}$$

Linear dynamical systems

For $n > 1$, if A were diagonal, i.e.

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$$

Then

$$A^k = \begin{pmatrix} a_1^k & & & \\ & a_2^k & & \\ & & \ddots & \\ & & & a_n^k \end{pmatrix}$$

and, likely that we can analyze behavior $x(k)$
but, most matrices are not diagonal

Linear dynamical systems

Diagonalization: for a large class of matrices A,

it can be represented as $A = S\Lambda S^{-1}$, where diagonal matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

and $S \in \mathbb{R}^{n\times n}$ is invertible matrix

Then

$$\begin{aligned} x(k) &= (S\Lambda S^{-1})^k x(0) \\ &= S\Lambda^k S^{-1} x(0) = S\Lambda^k c \end{aligned}$$

where $c = c(x(0)) = S^{-1}x(0) \in \mathbb{R}^n$

Linear dynamical systems

Suppose

$$S = \begin{pmatrix} | & & | \\ s_1 & \dots & s_n \\ | & & | \end{pmatrix}$$

Then

$$\begin{aligned} x(k) &= S\Lambda^k c \\ &= \sum_{i=1}^{n} c_i \lambda_i^k s_i \end{aligned}$$

Linear dynamical systems

Let $0 \leq |\lambda_n| \leq |\lambda_{n \quad 1}| \leq \cdots \leq |\lambda_2| < |\lambda_1|$

$$x(k) = \sum_{i=1}^{n} c_i \lambda_i^k s_i \;=\; \lambda_1^k \Big(c_1 s_1 + \sum_{i=2}^{n} c_i \big(\frac{\lambda_i}{\lambda_1}\big)^k s_i \Big)$$

Then

$$\|x(k)\| \stackrel{k\to\infty}{\to} \begin{cases} 0 \text{ if } |\lambda_1| < 1 \\ |c_1| \|s_1\| \text{ if } |\lambda_1| = 1 \\ \infty \text{ if } |\lambda_1| > 1 \end{cases}$$

moreover, for $|\lambda_1| > 1$,

$$\|\lambda_1^{\quad k} x(k) - c_1 s_1\| \to 0.$$

Diagonalization

When can a matrix $A \in \mathbb{R}^{n \times n}$ be diagonalize?

- When A has n distinct eigenvalues, for example
- In general, all matrices are block-diagonalizable a la Jordan form

Eigenvalues of A

- Roots of n order (characteristic) polynomial: $\det(A - \lambda I) = 0$
- Let they be $\lambda_1, \ldots, \lambda_n$

Eigenvectors of A

- Given λ_i, let $s_i \neq \mathbf{0}$ be such that $A s_i = \lambda_i s_i$
- Then s_i is eigenvector corresponding to eigenvalue λ_i

If all eigenvalues are distinct, then

- eigenvectors are linearly independent

Diagonalization

If all eigenvalues are distinct, then
eigenvectors are linearly independent

Proof. Suppose not and let s_1, s_2 are linearly dependent.
that is, $a_1 s_1 + a_2 s_2 = \mathbf{0}$ for some $a_1, a_2 \neq 0$
that is, $a_1 A s_1 + a_2 A s_2 = \mathbf{0}$, and hence $a_1 \lambda_1 s_1 + a_2 \lambda_2 s_2 = \mathbf{0}$
multiplying first equation by λ_2 and subtracting second

$$a_1(\lambda_2 - \lambda_1) s_1 = \mathbf{0}$$

that is, $a_1 = 0$; similarly, $a_2 = 0$. Contradiction.
argument can be similarly extended for case of n vectors.

Diagonalization

If all eigenvalues are distinct ($\lambda_i \neq \lambda_j,\ i \neq j$), then eigenvectors, $s_1, \dots, s_n$, are linearly independent

Therefore, we have invertible matrix S, where

$$S = \begin{pmatrix} | & & | \\ s_1 & \dots & s_n \\ | & & | \end{pmatrix}$$

Consider diagonal matrix of eigenvalues

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

Diagonalization

Consider

$$AS = \begin{pmatrix} | & & | \\ \lambda_1 s_1 & \dots & \lambda_n s_n \\ | & & | \end{pmatrix}$$
$$= \begin{pmatrix} | & & | \\ s_1 & \dots & s_n \\ | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$
$$= S\Lambda$$

Therefore, we have diagonalization $A = S\Lambda S^{-1}$

Remember: not every matrix is diagonalizable, e.g. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Linear dynamical systems

Let us consider linear system with $b \neq \mathbf{0}$:

$$
\begin{aligned}
x(k+1) &= Ax(k) + b \\
&= A(Ax(k-1) + b) + b \;=\; A^2 x(k-1) + \;(A+I)b \\
&\dots \\
&= A^k x(0) + \Big(\sum_{j=0}^{k\,-1} A^{k-j\;-1} \Big) b.
\end{aligned}
$$

Let $A = S\Lambda S^{-1}$, $c = S^{-1}x(0)$ and $d = S^{-1}b$. Then

$$
x(k+1) = \sum_{i=1}^{n} c_i s_i \lambda_i^k + d_i s_i \Big(\sum_{j=0}^{k-1} \lambda_i^j \Big)
$$

Linear dynamical systems

Let $A = S\Lambda S^{-1}$, $c = S^{-1}x(0)$ and $d = S^{-1}b$. Then

$$x(k+1) = \sum_{i=1}^{n} c_i s_i \lambda_i^k + d_i s_i (\sum_{j=0}^{k\quad 1} \lambda_i^j)$$

Let $0 \leq |\lambda_n| \leq |\lambda_{n\quad 1}| \leq \cdots \leq |\lambda_2| \leq |\lambda_1|$. Then

- If $|\lambda_1| \quad 1$, the sequence is divergent ($\rightarrow \infty$)
- If $|\lambda_1| < 1$, it converges as

$$x(k) \overset{k\rightarrow\infty}{\rightarrow} \sum_{i=1}^{n} s_i \frac{d_i}{1-\lambda_i}$$

$$= S \begin{pmatrix} \frac{1}{1-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{1-\lambda_n} \end{pmatrix} S^{-1} b = (I - A)^{-1} b$$

Linear dynamical systems

For linear system, equilibrium $x^\star$ should satisfy

$$x^\star = Ax^\star + b$$

The solution to the above exists when A does not have an eigenvalue equal to 1, which is

$$x^\star = (I - A)^{-1} b$$

But, as discussed, it may not be reached unless $|\lambda_1| < 1$!

Nonlinear dynamical systems

Consider nonlinear system

$$\begin{aligned} x(k+1) &= F(x(k)) \\ &= x(k) + \left(F(x(k)) - x(k)\right) \\ &= x(k) + G(x(k)) \end{aligned}$$

where $G(x) = F(x) - x$

Continuous approximation of the above (replace k by time index t)

$$\frac{dx(t)}{dt} = G(x(t))$$

When does $x(t) \to x^\star$?

Lyapunov function

Let there exist a Lyapunov (or Energy) function $V : \mathbb{R}^n \to \mathbb{R}_+$

Such that

1. V is minimum at $x^\star$
2. $\frac{dV(x(t))}{dt} < 0$ if $x(t) \neq x^\star$

   that is, $\nabla V(x(t))^T G(x(t)) < 0$ if $x(t) \neq x^\star$

Then $x(t) \to x^\star$

Lyapunov function: An Example

A simple model of Epidemic

Let $I(k) \in [0,1]$ be fraction of population that is infected
and $S(k) \in [0,1]$ be the fraction of population that is susceptible to infection
Population is either infected or susceptible: $I(k) + S(k) = 1$

Due to "social interaction" they evolve as

$$I(k+1) = I(k) + \beta I(k)S(k)$$
$$S(k+1) = S(k) - \beta I(k)S(k)$$

where $\beta \in (0,1)$ is a parameter captures "infectiousness"

Question: what is the equilibrium of such a society?

Lyapunov function: An Example

Since $I(k) + S(k) = 1$, we can focus only on one of them, say $S(k)$

Then

$$S(k+1) = S(k) - \beta(1 - S(k))S(k)$$

That is, continuous approximation suggests

$$\frac{dS(t)}{dt} = -\beta(1 - S(t))S(t).$$

An easy Lyapunov function is $V(S) = S$

Lyapunov function: An Example

For $V(S) = S$:

$$\frac{dV(S(t))}{dt} = V'(S(t))\frac{dS(t)}{dt}$$
$$= \beta(1 - S(t))S(t)$$

Then, for $S(t) \in [0, 1)$ if $S(t) \neq 0$,

$$\frac{dV(S(t))}{dt} < 0$$

And V is minimized at 0

Therefore, if $S(0) < 1$, then $S(t) \to 0$: entire population is *infected*!

Positive linear system

Positive linear system

Let $A = [A_{ij}] \in \mathbb{R}^{n \times n}$ be such that $A_{ij} > 0$ for all $1 \leq i, j \leq n$
System dynamics:

$$x(k) = Ax(k-1), \quad \text{for} \quad k \geq 1.$$

Perron-Frobenius Theorem: let $A \in \mathbb{R}^{n \times n}$ be positive

Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues such that

$$0 \leq |\lambda_n| \leq |\lambda_{n \; 1}| \leq \cdots \leq |\lambda_2| \leq |\lambda_1|$$

Then, maximum eigenvalue $\lambda_1 > 0$
It is unique, i.e. $|\lambda_1| > |\lambda_2|$
Corresponding eigenvector, say s_1 is component-wise > 0

Perron Frobenius Theorem

- Why is $\lambda_1 > 0$?
  - Let $\mathbb{T} = \{t > 0 : Ax \geq tx, \text{ for some } x \in \mathbb{R}^n_+\}$
  - $\mathbb{T}$ is non-empty: $A_{\text{min}} = \min_{ij} A_{ij} \in \mathbb{T}$ because
    - $A\mathbf{1} \geq A_{\text{min}}\mathbf{1}$ and $A_{\text{min}} > 0$ since $A > 0$
  - $\mathbb{T}$ is bounded because
    - $Ax \not\geq nA_{\text{max}}x$ for any $x \in \mathbb{R}^n_+$, where $A_{\text{max}} = \max_{ij} A_{ij}$
  - Let $t^\star = \max\{t : t \in \mathbb{T}\}$
  - That is, there exists $x \in \mathbb{R}^n_+$ such that $Ax \geq t^\star x$
    - In fact, it must be $Ax = t^\star x$.
    - Because, if $Ax \geq t^\star x$, then $A^2 x > t^\star Ax$ because $A > 0$
    - This will contradict $t^\star$ being maximum over $\mathbb{T}$
  - For any eigenvalue, eigenvector pair (λ, z), i.e. $Az = \lambda z$
    - $|\lambda||z| = |Az| \leq A|z|$
    - therefore, $|\lambda| \leq t^\star$
  - Thus, we have established that eigenvalue with largest norm is $t^\star > 0$.

Perron Frobenius Theorem

Why is eigenvector $s_1 > 0$?

- By previous argument, $s_1 \in \mathbb{R}^n_+$ and hence non-negative components
- Now As_1 has all component > 0 since $A > 0$ and $s_1 \neq 0$
- And $As_1 = \lambda_1 s_1$. That is, all components of s_1 must be > 0

Why is $|\lambda_2| < \lambda_1$?

- Suppose $|\lambda_2| = \lambda_1$
- Then, we will argue that it is possible only if $\lambda_2 = \lambda_1$
- If so, we will find contradiction

Perron Frobenius Theorem

If $|\lambda_2| = \lambda_1 > 0$, then $\lambda_2 = \lambda_1$

- Let $r = \lambda_1 = |\lambda_2| > 0$
- Suppose $\lambda_2 \neq r$. That is either it is real with value r or complex
- Then there exists m 1 such that λ_2^m has negative real part
- Let $2\epsilon > 0$ be smallest diagonal entry of A^m
- Consider matrix $T = A^m - \epsilon I$, which by construction is positive
  - $\lambda_2^m - \epsilon$ is its eigenvalue
  - Since λ_2^m has negative real part: $|\lambda_2^m - \epsilon| > r^m$
  - That is, maximum norm of eigenvalue of T is $> r^m$
- A^m has eigenvalues λ_i^m, $1 \leq i \leq n$
  - It's eigenvalue with largest norm is r^m
- By construction, $T \leq A^m$ and both are positive. Therefore
  - $T^k \leq (A^m)^k$ and hence $\lim_{k\to\infty} \|T^k\|_F^{1/k} \leq \lim_{k\to\infty} \|A^{mk}\|_F^{1/k}$
  - Gelfand formula: for any matrix M, max norm of eigenvalues is equal
  - to $\lim_{k\to\infty} \|M^k\|^{1/k}$
  - A contradiction: max norm of evs of A^m is $r^m < |\lambda_2^m - \epsilon|$ for T!

Perron Frobenius Theorem

$\lambda_2 = \lambda_1 = r > 0$ is not possible

- Suppose $s_2 \neq s_1$ and $As_1 = rs_1$ and $As_2 = rs_2$
- We had argued that $s_1 > 0$
- $s_2 \neq 0$ is real valued (since null space of $A - rI$ is real valued) At least one component of s_2 is > 0 (else choose $s_2 = -s_2$)
- Choose largest $\alpha > 0$ so that $u = s_1 - \alpha s_2$ is non-negative
- By construction u must have at least one component 0 (else choose larger α!)
- And $Au = ru$
- That is not possible since $Au > 0$ and u has at least one zero component
- That is, we can not choose s_2 and hence λ_2 can not be equal to λ_1

Positive linear system

More generally, we call A positive system if

- $A \geq 0$ component-wise
  - For some integer $m \geq 1$, $A^m > 0$
  - If eigenvalues of A are λ_i, $1 \leq i \leq n$
  - Then eigenvalues of A^m are λ_i^m, $1 \leq i \leq m$
  - The Perron-Frobenius for A^m implies similar conclusions for A

Special case of positive systems are Markov chains

- we consider them next
- as an important example, we'll consider random walks on graphs

An Example

Shuffling cards

A special case of *Overhead* shuffle:

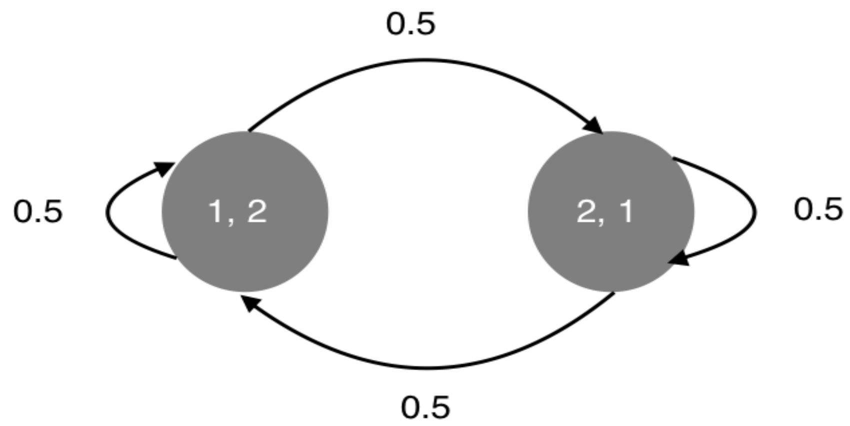
choose a card at random from deck and place it on top

How long does it take for card deck to become random?

Any one of 52! orderings of cards is equally likely

An Example

Markov chain for deck of 2 cards



Two possible card order: $(1,2)$ or $(2,1)$

Let X_k denote order of cards at time $k \geq 0$

$$\begin{aligned}\mathbb{P}(X_{k+1} = (1,2)) &= \mathbb{P}(X_k = (1,2) \text{ and card 1 chosen}) + \\ &\quad \mathbb{P}(X_k = (2,1) \text{ and card 1 chosen}) \\ &= \mathbb{P}(X_k = (1,2)) \times 0.5 + \mathbb{P}(X_k = (2,1)) \times 0.5 \\ &= 0.5\end{aligned}$$

Notations

Markov chain defined over state space $N = \{1, \dots, n\}$

$X_k \in N$ denote random variable representing state at ti me $k \geq 0$
$P_{ij} = \mathbb{P}(X_{k+1} = j | X_k = i)$ for all $i, j \in N$ and all $k \geq 0$

$$\mathbb{P}(X_{k+1} = i) = \sum_{j \in N} P_{ji} \mathbb{P}(X_k = j)$$

Let $p(k) = [p_i(k)] \in [0, 1]^n$, where $p_i(k) = \mathbb{P}(X_k = i)$

$$p_i(k+1) = \sum_{j \in N} p_j(k) P_{ji}, \ \forall i \in N \quad \Leftrightarrow \quad p(k+1)^T = p(k)^T P$$

$P = [P_{ij}]$: probability transition matrix of Markov chain

non-negative: $P \geq 0$
row-stochastic: $\sum_{j \in N} P_{ij} = 1$ for all $i \in N$

Stationary distribution

Markov chain dynamics: $p(k+1) = P^T p(k)$

- Let $P > 0$ ($P^T > 0$): positive linear system
- Perron-Frobenius: P^T has unique largest eigenvalue: $\lambda_{\max} > 0$
- Let $p^\star > 0$ be corresponding eigenvector: $P^T p^\star = \lambda_{\max} p^\star$

We claim $\lambda_{\max} = 1$ and $p(k) \to p^\star$

- Recall, $\|p(k)\| \to 0$ if $\lambda_{\max} < 1$ or $\|p(k)\| \to \infty$ if $\lambda_{\max} > 1$
- But $\sum_i p_i(k) = 1$ for all k, since $\sum_i p_i(0) = 1$ and

$$\begin{aligned}\sum_i p_i(k+1) &= p(k+1)^T \mathbf{1} = p(k)^T P \mathbf{1} \\ &= \sum_{ij} p_i(k) P_{ij} = \sum_i p_i(k) \sum_j P_{ij} \\ &= \sum_i p_i(k).\end{aligned}$$

Therefore, $\lambda_{\max}$ must be 1 and $p(k) \to p^\star$ (as argued before)

Stationary distribution

Stationary distribution: if $P > 0$, then there exists $p^\star > 0$ such that

$$p^\star = P^T p^\star \quad \Leftrightarrow \quad p_i^\star = \sum_j P_{ji} p_j^\star, \ \forall i.$$

$$p(k) \overset{k\to\infty}{\to} p^\star$$

- More generally, above holds when $P^k > 0$ for some $k \geq 1$
  - Sufficient *structural* condition: P is irreducible and aperiodic
  - Irreducibility
    - for each $i \neq j$, there is a positive probability to reach j starting from i
  - Aperiodicity
    - There is no partition of N so that Markov chain state 'periodically' rotates through those partitions
    - Special case: for each i, $P_{ii} > 0$

Stationary distribution

Reversible Markov chain with transition matrix $P \geq 0$

There exists $q = [q_i] > 0$ such

$$q_i P_{ij} = q_j P_{ji}, \quad \forall \quad i \neq j \in N \tag{3}$$

Then, stationary distribution, $p^\star$ exists such that

$$p^\star = \frac{1}{(\sum_i q_i)} q$$

Because, by (3) and P being stochastic

$$\begin{aligned} \sum_j q_j P_{ji} &= \sum_j q_i P_{ij} \\ &= q_i (\sum_j P_{ij}) \\ &= q_i \end{aligned}$$

Random walk on Graph

Consider an undirected connected graph G over $N = \{1, \ldots, n\}$

It's adjacency matrix A

Let k_i be degree of node $i \in N$

Random walk on G

Each time, remain at current node or walk to a random neighbor

Precisely, for any $i, j \in N$

$$P_{ij} = \begin{cases} \frac{1}{2} \text{ if } i = j \\ \frac{1}{2k_i} \text{ if } A_{ij} > 0, i \neq j \\ 0 \text{ if } A_{ij} = 0, i \neq j \end{cases}$$

Does it have stationary distribution? If yes, what is it?

Random walk on Graph

Answer: Yes, because irreducible and aperiodic.

Further, $p_i^\star = k_i/2m$, where m is number of edges

Why? (alternative approach: reversible MC)

$P = \frac{1}{2}(I + D^{\ 1}A)$, $p^\star = \frac{1}{2m}D\mathbf{1}$, where $D = diag(k_i)$, $\mathbf{1} = [1]$

$$
\begin{aligned}
p^{\star,T} P &= \frac{1}{2} p^{\star,T} (I + D^{-1}A) = \frac{1}{2} p^{\star,T} + \frac{1}{2} p^{\star,T} D^{-1} A \\
&= \frac{1}{2} p^{\star,T} + \frac{1}{2m} \mathbf{1}^T A \\
&= \frac{1}{2} p^{\star,T} + \frac{1}{4m} (A\mathbf{1})^T, \quad \text{because} \quad A = A^T \\
&= \frac{1}{2} p^{\star,T} + \frac{1}{4m} [k_i]^T = \frac{1}{2} p^{\star,T} + \frac{1}{2} p^{\star,T} = p^{\star,T}.
\end{aligned}
$$

Eigenvector Centrality

Stationary distribution of random walk:

$p^\star = \frac{1}{2}(I + D^{-1}A)p^\star$

$p_i^\star \propto k_i \rightarrow$ *Degree centrality*!

Eigenvector centrality (Bonacich '87)

Given (weighted, non-negative) adjacency matrix A associated with graph G

$\mathbf{v} = [v_i]$ be eigenvector associated with largest eigenvalue $\kappa > 0$

$$A\mathbf{v} = \kappa\mathbf{v} \quad \Leftrightarrow \quad \mathbf{v} = \kappa^{-1}A\mathbf{v}$$

Then v_i is eigenvector centrality of node $i \in N$

$$v_i = \kappa^{-1}\sum_j A_{ij}v_j$$

Katz Centrality

More generally (Katz '53):

Consider solution of equation

$$\mathbf{v} = \alpha A \mathbf{v} + \boldsymbol{\beta}$$

for some $\alpha > 0$ and $\boldsymbol{\beta} \in \mathbb{R}^{\mathbf{n}}$

Then v_i is called Katz centrality of node i

Recall

Solution exists if

$\det(I - \alpha A) \neq 0$

equivalently A doesn't have α^{-1} as eigenvalue

But dynamically solution is achieved if

largest eigenvalue of A is smaller than α^{-1}

Dynamic range of interest: $0 < \alpha < \lambda_{\max}^{-1}(A)$

PageRank

Goal: assign "importance" to each web-page in WWW

Utilize it to order pages for providing most relevant search results

An insight

If a page is important, and it points to other page, it must be important

But the influence of a page should not amplify with number of neighbors

Formalizing the insight: v_i be importance of page i

$$v_i = \alpha \sum_j A_{ij} v_j / k_j + \beta,$$

for some $\alpha > 0$ and $\beta \in \mathbb{R}$

PageRank

PageRank vector **v** is solution of

$$\mathbf{v} = \alpha A D^{-1} \mathbf{v} + \beta \mathbf{1},$$

where $D = diag(k_i)$ and **1** is vector of all 1

Solution

$$\begin{aligned} \mathbf{v} &= \beta (I - \alpha A D^{-1})^{-1} \mathbf{1} \\ &= \beta (I + \alpha A D^{-1} + \alpha^2 (A D^{-1})^2 + \ldots) \mathbf{1} \end{aligned}$$

That is, PageRank of page i is

sum of weighted paths in it's neighborhood plus a constant

Hubs and Authority

Goal: assign importance to authors

By utilizing whose papers are cited by whom

An additional insight

A node is important if it points to other important node

For example, a review article is useful if it points to important works

Two types of important nodes

Authorities: nodes that are important due to having useful information

Hubs: nodes that tell where important authorities are

HITS: Hyperlink induced topic search

Formalizing

x_i and y_i be authority and hub centrality of node i

x_i is high if it is *pointed* to by hubs with high centrality

$$x_i = \alpha \sum_j A_{ij} y_j, \quad \text{for some} \quad \alpha > 0$$

y_i is high if it *points* to authorities with high centrality

$$y_i = \beta \sum_j A_{ji} x_j \quad \text{for some} \quad \beta > 0$$

HITS: Hyperlink induced topic search

Summarizing

$$\mathbf{x} = \alpha A\mathbf{y}$$
$$\mathbf{y} = \beta A^T\mathbf{x}.$$

Therefore (with $\lambda = (\alpha\beta)^{-1}$)

$$\mathbf{x} = \alpha\beta AA^T\mathbf{x} \quad \Leftrightarrow \quad AA^T\mathbf{x} = \lambda\mathbf{x}$$
$$\mathbf{y} = \alpha\beta A^T A\mathbf{y} \quad \Leftrightarrow \quad A^T A\mathbf{y} = \lambda\mathbf{y}.$$

HITS algorithm

Solve for $\mathbf{x}$ in $AA^T\mathbf{x} = \lambda\mathbf{x}$ for largest eigenvector $\lambda > 0$
Recover $\mathbf{y} = A^T\mathbf{x}$

HITS: Hyperlink induced topic search

HITS algorithm: for each $i \in N$

$$x_i = \kappa \sum_j (AA^T)_{ij} x_j$$
$$y_i = \kappa \sum_j (A^T A)_{ij} y_j .$$

$(AA^T)_{ij} = \sum_k A_{ik} A_{jk}$

Shared citations for i and j

Important authorities are cited by (many) others together

$(A^T A)_{ij} = \sum_k A_{ki} A_{ki}$

Shared references of i and j

Important hubs refer to (many) others together