14.30 PROBLEM SET 9 SUGGESTED ANSWERS

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Problem 1

For parts (**a**)-(**g**), please refer to the relevant sections in the handouts/textbooks.

(h) False. With the knowledge of the distribution of a statistic under the null hypothesis, we can calculate only α, but not β.

(i) False. In particular, there are hypothesis tests for which one can form a GLRT where no optimal test exists, so GLRT could not be optimal in that case.

(j) Yes. Hypothesis tests are typically constructed by using a test statistic T. The null hypothesis is rejected if T lies in some interval or if T lies outside of some interval. The interval is chosen to make the test have a desired significance level. This procedure is in essence the same as the construction of the confidence interval.

(k) Recall that α and β are calculated under different distributions (under the null and under the alternative hypothesis). There is no reason that it always must be $\alpha + \beta = 1$, although it is possible under some special circumstances. Also, *in general,* it is not possible $\alpha = \beta = 0$, again in some cases we might have $\alpha = 0$ or $\beta = 0$, but in many cases neither is possible.

NOTE: We might have some cases that $\alpha = \beta = 0$ under very special and unrealistic situation (can you think of an example of such case ?), and in that case, the hypothesis test becomes trivial and uninteresting. (why ?)

Problem 2

(a) The probability of committing a Type I error is calculated as follows:

$$\begin{aligned} P(\text{Type I error}) &= P(\text{reject } H_0 | H_0 \text{ is true}) \\ &= P(Y \geq 3.20 | \lambda = 1) \\ &= \int_{3.20}^{\infty} e^{-y} dy \\ &= 0.04 \end{aligned}$$

(b) The probability of committing a Type II error when $\lambda = 4/3$ is calculated as follows:

$$\begin{aligned}
P(\text{Type II error}) &= P(\text{don't reject } H_0|H_1 \text{ is true}) \\
&= P(Y \leq 3.20|\lambda = \frac{4}{3}) \\
&= \int_0^{3.20} \frac{3}{4} e^{-3y/4} dy \\
&= \int_0^{2.4} e^{-u} du \text{ (change in variable: } u = \frac{3}{4}y) \\
&= 0.91
\end{aligned}$$

Problem 3

(a) i) We first find formulas for α and β in terms of k:

$$\begin{aligned}
\alpha &= P(\text{Type I error}) = P(\text{reject } H_0|H_0 \text{ is true}) \\
&= \int_0^k f_0(x)dx = \int_0^k 2xdx \\
&= k^2
\end{aligned}$$

$$\begin{aligned}
\beta &= P(\text{Type II error}) = P(\text{don't reject } H_0|H_A \text{ is true}) \\
&= \int_k^1 f_A(x)dx = \int_k^1 (2-2x)dx \\
&= k^2 - 2k + 1
\end{aligned}$$

Note that we were able to write down expressions for α and β because with only 1 observation x, we knew that x had to be our statistic, and furthermore our test would be of the form "reject H_0 for $x < k$" because we are more likely to have observed a small x if H_A is true.

$$\begin{aligned}
\min_k(\alpha + \beta) &= \min_k(2k^2 - 2k + 1) \\
\frac{\partial(2k^2 - 2k + 1)}{\partial k} &= 4k - 2 = 0
\end{aligned}$$

So $\alpha + \beta$ is minimized at $k = 1/2$.

ii) $\alpha + \beta = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

iii) With $k = 1/2$, the testing procedure is then to reject the null for $x < k = 1/2$. So we do not reject the null for $x = 0.6$

(b) i) Find k such that

$$\int_0^k f_0(x)dx = \int_0^k 2xdx = k^2 = 0.1 \quad \text{[why 0.1, when the restriction is } \alpha \leq 0.10 \text{ ?]}$$

this implies $k = \sqrt{0.1}$.

ii) Then

$$\begin{aligned}\beta &= \int_{\sqrt{0.1}}^{1}(2-2x)dx \\ &= \left[2x-x^2\right]_{\sqrt{0.1}}^{1} \\ &= 1.1-2\sqrt{0.1}\end{aligned}$$

iii) For $x = 0.4$, we do not reject because

$$x = 0.4 > k = \sqrt{0.1}$$

(c) With ten observations, you could use, for instance

$$T(X) = \overline{X}$$

and reject the if $\overline{X} < k$ (for appropriate k).

Problem 4

We have two simple hypotheses, so we should be able to derive an optimal test statistic using the Neyman-Pearson lemma. The lemma tells us that the best test can be achieved by constructing the test statistic $T = \frac{f_1(x_1,...,x_n|\mu=1)}{f_0(x_1,...,x_n|\mu=0)}$ and then rejecting the null ($\mu = 0$) whenever $T > k$, where k is the critical value such that the probability of type I error is equal to $\alpha = 0.005$. We know that the likelihood function is

$$\begin{aligned}f(x_1,...,x_n \mid \mu) &= \prod_{i=1}^{n} f(x_i \mid \mu) \\ &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i-\mu)^2}{2}} \\ &= (2\pi)^{-\frac{n}{2}}\, e^{-\frac{1}{2}\sum(x_i-\mu)^2}\end{aligned}$$

Then we have

$$\begin{aligned}T &= \frac{(2\pi)^{-\frac{n}{2}}\, e^{-\frac{1}{2}\sum(x_i-1)^2}}{(2\pi)^{-\frac{n}{2}}\, e^{-\frac{1}{2}\sum x_i^2}} \\ &= e^{-\frac{1}{2}\sum(-2x_i+1)} \\ &= e^{n\overline{X}-\frac{n}{2}}\end{aligned}$$

Thus we should reject the null hypothesis if $e^{n\overline{X}-\frac{n}{2}} > k$, or equivalently, reject the null if $\overline{X} > \frac{2\ln(k)+n}{2n}$. Thus we have a test of the form "reject if $\overline{X} > c$" for some $c(n)$, and we know how to determine the proper value of c for our desired confidence level. We want $\Pr\left(\overline{X} > c \mid \mu = 0\right) = \alpha = 0.005$, so we use

$$
\begin{aligned}
0.005 &= \Pr\left(\overline{X} > c \mid \mu = 0\right) \\
&= \Pr\left(\frac{\overline{X}-0}{\sqrt{\frac{1}{n}}} > \frac{c-0}{\sqrt{\frac{1}{n}}} \mid \mu = 0\right) \\
&= \Pr\left(Z > \sqrt{n}c\right)
\end{aligned}
$$

From the Z-table, I find that we must have $\sqrt{n}c = 1.65$, so $c = \frac{1.65}{\sqrt{n}}$. Thus the final form of our test is that we reject the null if $\overline{X} > \frac{1.65}{\sqrt{n}}$.

Problem 5

(a) $H_0 : \mu = 7 \;\; versus \;\; H_1 : \mu = 6.$

(b) The test statistics is sample mean: $T = \overline{X}$. And under the null $\overline{X} \sim N(7, 1/\sqrt{10})$,and under the alternative $\overline{X} \sim N(7, 1\sqrt{10})$. We reject H_0 if

$$
\begin{aligned}
\overline{X} &< 7 - \frac{1}{\sqrt{10}}\Phi^{-1}(0.05) \\
&= 7 - \frac{1}{\sqrt{10}}(1.65) \\
&= 7 - (0.32)(1.65) \\
&= 6.47
\end{aligned}
$$

So we reject H_0 because $\overline{X} = 6.2 < 6.47$.

(c)

$$
\begin{aligned}
Power &= 1 - \beta \\
&= 1 - P(\text{don't reject } H_0 | H_A \text{ is true}) \\
&= 1 - \left[1 - \Phi\left(\frac{6.47-6}{1/\sqrt{10}}\right)\right] \\
&= \Phi\left(\frac{6.47-6}{1/\sqrt{10}}\right) \\
&= \Phi\left(1.49\right) \\
&= 0.93
\end{aligned}
$$

(d) With unknown σ^2, the test statistic under the null is

$$
T = \frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t_{(10-1)}
$$

we reject H_0 if

$$\begin{aligned}
\frac{\overline{X}-\mu}{S/\sqrt{n}} &< t_9(\alpha), \text{ where } \alpha \text{ is a significance level} \\
&\Leftrightarrow \\
\frac{6.2-7}{\sqrt{1.5}/\sqrt{10}} &= -2.5 < -1.83 \text{ (from the table)}
\end{aligned}$$

So we reject H_0.

(e) In this case, the test then becomes two-sided [$H_0 : \mu = 7$ *versus* $H_1 : \mu \neq 7$], so all the procedures used before should be modified accordingly.

Problem 6

Suppose $X_i \sim N(\mu_X, 1), i = 1, ..., n_X$ and $Z_j \sim N(\mu_Z, 1), j = 1, ..., n_Z$ and all the observations are independent. You want to test the hypothesis that the means are equal against the alternative that they are not. Use the statistic

$$T = \frac{(\overline{X}-\overline{Z})}{\sqrt{1/n_X + 1/n_Z}}:$$

(a) The test statistic T can take any real number, and if the absolute value of T is 'sufficiently' different from 0, we will reject the null hypothesis that the means are equal.

(b) Note that T is a linear transformation of $(n_X + n_Z)$ normal random variables - hence, we can guess that $T \sim N(E[T], Var[T])$.The mean and variance of T are:

$$\begin{aligned}
E[T] &= E\left[\frac{(\overline{X}-\overline{Z})}{\sqrt{1/n_X + 1/n_Z}}\right] \\
&= \frac{1}{\sqrt{1/n_X + 1/n_Z}} E[(\overline{X}-\overline{Z})] \\
&= \frac{1}{\sqrt{1/n_X + 1/n_Z}} \left(E[\overline{X}] - E[\overline{Z}]\right) \\
&= \frac{1}{\sqrt{1/n_X + 1/n_Z}} \left(E\left[\frac{1}{n_X}\sum X_i\right] - E\left[\frac{1}{n_Z}\sum Z_j\right]\right) \\
&= \frac{1}{\sqrt{1/n_X + 1/n_Z}} (\mu_X - \mu_Z) \\
&= 0 \quad (\text{under the null} \mu_X = \mu_Z)
\end{aligned}$$

$$\begin{aligned}
Var[T] &= Var\left[\frac{(\overline{X}-\overline{Z})}{\sqrt{1/n_X+1/n_Z}}\right] \\
&= \left(\frac{1}{1/n_X+1/n_Z}\right) Var[\overline{X}-\overline{Z}] \\
&= \left(\frac{1}{1/n_X+1/n_Z}\right)\left(Var[\overline{X}]+Var[\overline{Z}]\right) \text{ (independence)} \\
&= \left(\frac{1}{1/n_X+1/n_Z}\right)\left(Var\left[\frac{1}{n_X}\sum X_i\right]+Var\left[\frac{1}{n_Z}\sum Z_j\right]\right) \\
&= \left(\frac{1}{1/n_X+1/n_Z}\right)\left(\frac{1}{n_X^2}n_X(1)+\frac{1}{n_Z^2}n_Z(1)\right) \\
&= \left(\frac{1}{\frac{n_X+n_Z}{n_Xn_Z}}\right)\left(\frac{1}{n_X}+\frac{1}{n_Z}\right) \\
&= \left(\frac{n_Xn_Z}{n_X+n_Z}\right)\left(\frac{n_X+n_Z}{n_Xn_Z}\right) \\
&= 1
\end{aligned}$$

So $T \sim N(0,1)$.

(c) Since the distribution of the test statistic T is the standard normal under the null, we reject the null when $|t| > z_{\alpha/2}$.

Problem 7

(a) Consider the test statistic $T = \frac{S_X^2}{S_Y^2}$, where $S_X^2 = \frac{1}{n_X-1}\sum_{i=1}^{n_X}\left(X_i-\overline{X}\right)^2$ and S_Y^2 is defined analogously. We know that $\frac{(n_X-1)S_X^2}{\sigma_X^2} \sim \chi^2_{(n_X-1)}$ and $\frac{(n_Y-1)S_Y^2}{\sigma_Y^2} \sim \chi^2_{(n_Y-1)}$, and that these two statistics are independent. Under the null hypothesis, $\sigma_X^2 = \sigma_Y^2$, so we can rewrite T as $\frac{\frac{(n_X-1)S_X^2}{\sigma_X^2(n_X-1)}}{\frac{(n_Y-1)S_Y^2}{\sigma_Y^2(n_Y-1)}} \sim F\left(n_X-1, n_Y-1\right)$. So we reject for $T > c$, where c is defined by $\Pr\left(F\left(n_X-1,n_Y-1\right)>c\right) = \alpha = 0.10$.

(b) We have $n_X = 6$, $n_Y = 4$, $\overline{X} = 12$, $\overline{Y} = 2.75$, $\sum_{i=1}^{n_X}\left(X_i-\overline{X}\right)^2 = 118$, and $\sum_{i=1}^{n_Y}\left(Y_i-\overline{Y}\right)^2 = 8.75$. So $T = \frac{118/5}{8.75/3} = 8.09$. From the F-table, our critical value, c, is equal to 5.31, so we reject the null hypothesis.