14.30 Introduction to Statistical Methods in Economics
Spring 2009

Problem Set #4 - Solutions

14.30 - Intro. to Statistical Methods in Economics

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Question One

Suppose that the PDF of X is as follows:

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}.$$

1. Determine the PDF for $Y = X^{\frac{1}{2}}$.

   - Solution to 1: In order to find the PDF, we can use the CDF or "2-Step" method. We write:

   $$\begin{aligned} F_Y(y) &= P(Y \leq y) = P(X^{\frac{1}{2}} \leq y) = \int_{x:x^{\frac{1}{2}} \leq y} f_X(x)dx \\ &= \int_{x:x \leq y^2} f_X(x)dx \\ &= \int_0^{y^2} e^{-x}dx \\ &= (-e^{-x})_0^{y^2} \\ F_Y(y) &= 1 - e^{-y^2} \\ f_Y(y) &= 2ye^{-y^2} \end{aligned}$$

   for $y > 0$ and zero for $y \leq 0$.

2. Determine the PDF for $W = X^{\frac{1}{k}}$ for $k \in \mathbb{N}$.

   - Solution to (2): This is just a straightforward generalization of part 1. We can write:

$$
\begin{aligned}
F_W(w) &= P(W \leq w) = P(X^{\frac{1}{k}} \leq w) = \int_{x:x^{\frac{1}{k}} \leq w} f_X(x)dx \\
&= \int_{x:x \leq w^k} f_X(x)dx \\
&= \int_0^{w^k} e^{-x}dx \\
&= (-e^{-x})_0^{w^k} \\
F_W(w) &= 1 - e^{-w^k} \\
f_W(w) &= kw^{k-1}e^{-w^k}
\end{aligned}
$$

for $w > 0$ and zero for $w \leq 0$.

Question Two

Suppose that the PDF of a random variable X is as follows:

$$
f(x) = \begin{cases} \frac{2}{25}x & \text{for } 0 < x < 5 \\ 0 & \text{otherwise} \end{cases}
$$

Also, suppose that $Y \equiv X(5-X)$. Determine the PDF and CDF of Y. You can solve this in two ways. First, you can compute $f_Y(y)$ using the formula given in class:

$$
f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|,
$$

taking care that $g(x)$ is piece-wise monotonic. Second, you can solve this by finding $F_Y(y) = P[Y \leq y]$ directly, as we did in recitation. You will receive extra-credit if you can do it both ways.

- Solution: We first need to find the inverse function, $g^{-1}(y) = x$. By solving we obtain:

$$
\begin{aligned}
Y &= X(5 - X) \\
0 &= -X^2 + 5X - Y \\
X &= \frac{5 \pm \sqrt{25 - 4Y}}{2}
\end{aligned}
$$

Now, we can apply the transformation result above since we do have a piecewise monotonic function, $g(x)$, with two roots over the interval. Since we know it is a parabola, we solve for where the derivative is zero in order to obtain the two monotonic pieces (one will be monotonically increasing, the other decreasing). So, we find

$$g'(x) = 5 - 2x = 0 \Rightarrow x = \frac{5}{2}.$$

So, it turns out that at the midpoint, we have a maximum (since the second derivative is negative).
We now simply apply the formula to the two halves of the function and add them together:

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= \frac{2}{25} \left(\frac{5 \pm \sqrt{25-4y}}{2} \right) \left| \frac{d}{dy} \frac{5 \pm \sqrt{25-4y}}{2} \right| \\
&= \frac{1}{50} \left(5 \pm \sqrt{25-4y} \right) \left| \frac{d}{dy} 5 \pm \sqrt{25-4y} \right| \\
&= \frac{1}{50} \left(5 \pm \sqrt{25-4y} \right) \left| \pm \frac{\frac{1}{2} \cdot -4}{\sqrt{25-4y}} \right| \\
f_Y(y) &= \begin{cases} \frac{1}{25} \left(\frac{5}{\sqrt{25-4y}} + 1 \right) & \text{if } 0 < y \leq \frac{25}{4} \\ \frac{1}{25} \left(\frac{5}{\sqrt{25-4y}} - 1 \right) & \text{if } 0 < y \leq \frac{25}{4} \end{cases} \\
&= \frac{1}{25} \left(\frac{5}{\sqrt{25-4y}} + 1 \right) + \frac{1}{25} \left(\frac{5}{\sqrt{25-4y}} - 1 \right) \\
f_Y(y) &= \frac{2}{5\sqrt{25-4y}}
\end{aligned}$$

To get the CDF, we just integrate:

$$\begin{aligned}
F_Y(y) &= \int_0^y \frac{2}{5\sqrt{25-4y'}} dy' \\
&= \left[\frac{2}{5} \cdot \left(-\frac{1}{4} \right) 2\sqrt{25-4y'} \right]_0^y \\
&= \left[-\frac{1}{5} \sqrt{25-4y'} \right]_0^y \\
&= \left[-\frac{1}{5} \sqrt{25-4y} + \frac{1}{5} \sqrt{25} \right] \\
F_Y(y) &= 1 - \frac{1}{5} \sqrt{25-4y}.
\end{aligned}$$

Both the PDF and CDF are defined on the interval $0 < y \leq \frac{25}{4}$ and the PDF is zero otherwise and the CDF is zero for $y \leq 0$ and one for $\frac{25}{4} < y$. Now, just to check our answer (and for extra credit), we will also use the CDF or "2-Step" method:

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(X(5-X) \leq y) = \int_{x:x(5-x)\leq y} f_X(x)dx \\
&= \int_{x:0\leq x\leq \frac{5-\sqrt{25-4y}}{2}} \frac{2}{25}xdx + \int_{x:\frac{5+\sqrt{25-4y}}{2}\leq x\leq 5} \frac{2}{25}xdx \\
&= \left(\frac{1}{25}x^2\right)_0^{\frac{5-\sqrt{25-4y}}{2}} + \left(\frac{1}{25}x^2\right)_{\frac{5+\sqrt{25-4y}}{2}}^{5} \\
&= \frac{1}{100}\left(5-\sqrt{25-4y}\right)^2 + 1 - \frac{1}{100}\left(5+\sqrt{25-4y}\right)^2 \\
&= 1 - \frac{1}{100}\left[\left(5-\sqrt{25-4y}\right)^2 - \left(5+\sqrt{25-4y}\right)^2\right] \\
&= 1 - \frac{1}{100}\left[\left(25-10\sqrt{25-4y}+25-4y\right) - \left(25+10\sqrt{25-4y}+25-4y\right)\right] \\
&= 1 - \frac{1}{100}\left(20\sqrt{25-4y}\right) \\
F_Y(y) &= 1 - \frac{1}{5}\sqrt{25-4y} \\
f_Y(y) &= \frac{2}{5\sqrt{25-4y}}
\end{aligned}$$

on the interval $0 < y \leq \frac{25}{4}$. We got the same answer! Great!

Question Three

(Bain/Engelhardt, p. 226)

(6 points) Let X be a random variable that is uniformly distributed on $[0,1]$ (i.e. $f(x) = 1$ on that interval and zero elsewhere). Use two techniques from class ("2-step"/CDF technique and the transformation method) to determine the PDF of each of the following:

1. $Y = X^{\frac{1}{4}}$.

   - Solution to (1): First, $g(x) = x^{\frac{1}{4}} \Rightarrow g^{-1}(y) = y^4$. Using the "2-step" technique, we get

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(X^{\frac{1}{4}} \leq y) = \int_{x:x^{\frac{1}{4}}\leq y} f_X(x)dx \\
&= \int_{x:x\leq y^4} dx \\
&= (x)_0^{y^4} \\
F_Y(y) &= y^4 \\
f_Y(y) &= 4y^3
\end{aligned}$$

Using the transformation technique (after checking that $g(x)$ is monotonic on the nonzero support of $f(x)$), we get

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y))\left|\frac{d}{dy}g^{-1}(y)\right| \\
&= f_X(y^4)\left|\frac{d}{dy}y^4\right| \\
&= 1\left|4y^3\right| \\
f_Y(y) &= 4y^3
\end{aligned}$$

where $f_Y(y)$ is defined above on $[0,1]$ and zero otherwise.

2. $W = e^{-X}$.

- Solution to (1): First, $g(x) = e^{-x} \Rightarrow g^{-1}(w) = -\log w$ (note: "log" typically denotes "ln" or the natural log, log base e in economics and many other sciences). Using the "2-step" technique while paying close attention to the inequalities, we get

$$\begin{aligned}
F_W(w) &= P(W \leq w) = P(e^{-x} \leq w) = \int_{x:e^{-x}\leq w} f_X(x)dx \\
&= \int_{x:x\geq -\log w} dx = \\
&= \int_{x:x\leq \log w} dx = \\
&= (x)_0^{\log w} \\
F_W(w) &= \log w \\
f_W(w) &= \frac{1}{w}
\end{aligned}$$

Using the transformation technique (after checking that $g(x)$ is monotonic on the nonzero support of $f(x)$), we get

$$\begin{aligned}
f_W(w) &= f_X(g^{-1}(w))\left|\frac{d}{dw}g^{-1}(w)\right| \\
&= f_X(-\log w)\left|\frac{d}{dw} - \log w\right| \\
&= 1\left|-\frac{1}{w}\right| \\
f_W(y) &= \frac{1}{w}
\end{aligned}$$

where $f_W(w)$ is defined above on $[\frac{1}{e},1]$ and zero otherwise.

3. $Z = 1 - e^{-X}$.

- Solution to (1): First, $g(x) = 1 - e^{-x} \Rightarrow g^{-1}(z) = -\log(1-z)$ (note: "log" typically denotes "ln" or the natural log, log base e in economics and many other sciences). Using the "2-step" technique, we get

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) = P(e^{-x} \leq z) = \int_{x:1-e^{-x}\leq z} f_X(x)dx \\
&= \int_{x:x\leq -\log(1-z)} dx \\
&= (x)_0^{-\log(1-z)} \\
F_Z(z) &= -\log(1-z) \\
f_Z(z) &= \frac{1}{1-z}
\end{aligned}$$

Using the transformation technique (after checking that $g(x)$ is monotonic on the nonzero support of $f(x)$), we get

$$\begin{aligned}
f_Z(z) &= f_X(g^{-1}(z)) \left| \frac{d}{dz} g^{-1}(z) \right| \\
&= f_X(-\log(1-z)) \left| \frac{d}{dw} - \log(1-z) \right| \\
&= 1 \left| \frac{1}{1-z} \right| \\
f_Z(z) &= \frac{1}{1-z}
\end{aligned}$$

where $f_Z(z)$ is defined above on $[0, 1 - \frac{1}{e}]$ and zero otherwise.

Question Four

(Bain/Engelhardt p. 227)

If $X \sim Binomial(n, p)$, then find the pdf of $Y = n - X$.

- Solution: The random variable $Y = n - X$ is a straightforward discrete transformation. We right the inverse function, $g^{-1}(y) = n - Y$. We now write the binomial pdf:

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

By inspection, we see that we can simply substitute in the linear transformation (which is monontonic with Jacobian is -1, i.e. absolute value of 1 for all possible outcomes):

$$\begin{aligned} f_Y(x) &= \binom{n}{n-y} p^{n-y}(1-p)^{n-(n-y)} \\ &= \binom{n}{y} p^{n-y}(1-p)^{y} \\ &= Binomial(n, 1-p) \end{aligned}$$

So, we see that this simple transformation simply relabeled a success as a failure and vice versa in our n Bernoulli trials. This is what we would have expected.

Question Five

(Bain/Engelhardt p. 227)

Let X and Y have joint PDF $f(x,y) = 4e^{-2(x+y)}$ for $0 < x < \infty$ and $0 < y < \infty$, and zero otherwise.

1. Find the CDF of $W = X + Y$.

   - Solution to (1): The CDF of $W = X + Y$ can be obtained by defining $Z = X$ and finding the joint distribution of W and Z, and then integrating out Z to obtain the marginal of W. We first define the transformation of x and y to obtain w and z and find its inverse:

$$\begin{aligned} g(x,y) &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (w,z) \\ g^{-1}(w,z) &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = (x,y) \end{aligned}$$

   The Jacobian is really easy to get once we've written $g(x,y)$ as a linear transformation in matrix notation:

$$J = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

   So, since $g(x,y)$ is linear (and, hence, monotonic), we can just use the transformation technique:

$$\begin{aligned} f(x,y) &= 4e^{-2(x+y)} \\ f(w,z) &= f(g^{-1}(w,z))\,|J| \\ &= f(z, w-z)\,|-1| \\ &= 4e^{-2(z+(w-z))} \\ &= 4e^{-2w} \end{aligned}$$

where $|\cdot|$ denotes the absolute value of the determinant operator. Now, to get the CDF we need to get the marginal of W and then integrate, taking into account the bounds on X and Y inducing bounds on W of $Z < W < \infty$:

$$\begin{aligned} f_W(w) &= \int_0^w f(w,z)dz \\ &= \int_0^w 4e^{-2w}dz \\ &= 4we^{-2w} \end{aligned}$$

Now that we have the marginal, we use integration by parts to obtain the CDF:

$$\begin{aligned} F_W(w) &= \int_0^w f_W(w')dw' \\ &= \int_0^w 4w'e^{-2w'}dw' \\ &= (-2w'e^{-2w'})_0^w - \int_0^w -2e^{-2w'}dw' \\ &= -2we^{-2w} - e^{-2w} + 1 \\ F_W(w) &= 1 - (2w+1)e^{-2w} \end{aligned}$$

Alternatively, we could have just used the convolution formula adapted to this problem:

$$f_W(w) = \int_0^\infty f(x, w-x)dx$$

which would have yielded the same solution:

$$\begin{aligned} f_W(w) &= \int_0^\infty f_X(x)f_Y(w-x)dx \\ &= \int_0^w 2e^{-2x} \cdot 2e^{-2(w-x)}dx \end{aligned}$$

which is the same integral we performed above.

2. Find the joint pdf of $U = \frac{X}{Y}$ and $V = X$.

- Solution to (2): We use similar methods to those in part (1). Define $g(x,y)$ and $g^{-1}(u,v)$:

$$\begin{aligned} g(x,y) &= (\frac{x}{y}, x) = (u,v) \\ g^{-1}(u,v) &= (v, \frac{v}{u}) = (x,y) \end{aligned}$$

with its corresponding Jacobian:

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{v}{u^2} & \frac{1}{u} \end{bmatrix}$$

which has a determinant of $|J| = \frac{v}{u^2}$. Since $x > 0$ and $y > 0$, we can use the transformation methods without worrying about multiple roots:

$$\begin{aligned} f(x,y) &= 4e^{-2(x+y)} \\ f(u,v) &= f(g^{-1}(u,v))\,|J| \\ &= f(v, \frac{v}{u})\frac{v}{u^2} \\ &= 4e^{-2(v+\frac{v}{u})}\frac{v}{u^2} \\ f(u,v) &= 4\frac{v}{u^2}e^{-2v\cdot\frac{u+1}{u}} \end{aligned}$$

So, we have obtained the joint pdf.

3. Find the marginal pdf of U.

- Solution to (3): The marginal pdf of U can be obtained by integrating out v:

$$\begin{aligned} \int_0^\infty f(u,v)dv &= \int_0^\infty 4\frac{v}{u^2}e^{-2v\cdot\frac{u+1}{u}}dv \\ &= \left(\frac{\frac{4}{u^2}v}{-2(1+\frac{1}{u})}e^{-2(1+\frac{1}{u})v}\right)_0^\infty - \frac{-4}{2u^2(1+\frac{1}{u})}\int_0^\infty e^{-2(1+\frac{1}{u})v}dv \\ &= \frac{2}{u^2(1+\frac{1}{u})}\left(\frac{1}{-2(1+\frac{1}{u})}e^{-2(1+\frac{1}{u})v}\right)_0^\infty \\ &= \frac{2}{u^2(1+\frac{1}{u})}\left(0 - \frac{1}{-2(1+\frac{1}{u})}\right) \\ f_U(u) &= \frac{1}{(u+1)^2} \end{aligned}$$

Finally, just to check to make sure that we have a valid PDF, we can integrate to verify that it does, in fact, integrate to one:

$$\begin{aligned} F_U(u) &= \int_0^\infty \frac{1}{(u+1)^2}du \\ &= 0 - \frac{-1}{0+1} = 1 \end{aligned}$$