14.381: Statistics

Problem Set 2

You should hand in the solution for problem 4. Problems 1-3 are for practice. One of the problems from the problem sets will be on the mid-term exam.

1. Let $X_1, X_2, \ldots, X_n$ be iid observations. Find minimal sufficient statistics

   (a) $f(x \mid \theta) = \frac{2x}{\theta^2}$, $0 < x < \theta$, $\theta > 0$;

   (b) $f(x \mid \theta) = e^{-(x-\theta)} \cdot \exp\left\{-e^{-(x-\theta)}\right\}$, $-\infty < x < \infty$, $-\infty < \theta < \infty$;

   (c) $f(x \mid \theta) = \frac{2!}{x!(2-x)!}\theta^x(1-\theta)^{2-x}$, $x \in \{0, 1, 2\}$, $0 \le \theta \le 1$.

2. Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with parameter λ

$$P\{X = j\} = \frac{e^{-\lambda}\lambda^j}{j!} \quad j = 0, 1, \ldots$$

   (a) Find a minimal sufficient statistic.

   (b) Assume that we are interested in estimating probability of a count of zero $\theta = P\{X = 0\} = \exp\{-\lambda\}$. Find an unbiased estimator of θ. *Hint:* $\theta = P\{X = 0\} = E\mathbb{I}\{X = 0\}$.

   (c) Is the estimator in (b) a function of a minimal sufficient statistics? Modify the estimator to make sure it is a function of a minimal sufficient statistics, while it is still unbiased. You may try to do analytical derivation (it can be done here). However, if it is too hard, then explain a Monte-Carlo procedure that you may use instead.

3. Assume $X_1, \ldots, X_n$ are iid with mean μ and variance σ^2 (both unknown). Let us estimate mean by

$$\hat{\mu} = \sum_{i=1}^{n} \omega_i X_i$$

(i) Under what condition is $\hat{\mu}$ unbiased?

(ii) Among all unbiased $\hat{\mu}$ find the one with the smallest variance.

(iii) What $\{\omega_i\}$ whould lead to the smallest MSE?

4. *(Required problem)* Suppose that the random variables $Y_1, ..., Y_n$ satisfy

$$Y_i = \beta x_i + e_i, i = 1, ..., n,$$

where $x_1, ..., x_n$ are fixed constants and $e_1, ..., e_n$ are i.i.d. normals with mean 0 and variance σ^2 (variance is unknown).

(a) Find a two-dimensional sufficient statistic for (β, σ^2).

(b) Find the MLE of β and show that it is unbiased.

(c) Find the distribution of the MLE of β.

(d) Is $\hat{\beta}_1 = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n x_i}$ an unbiased estimator for β? Find its variance.

(e) Is $\hat{\beta}_2 = \frac{1}{n}\sum_{i=1}^n \frac{Y_i}{x_i}$ an unbiased estimator for β? Find its variance.

(f) Which of the three estimator $\hat{\beta}_{MLE}, \hat{\beta}_1$ and $\hat{\beta}_2$ has the smallest variance? *Hint: you may need the following inequalities. For any numbers* $a_1, .., a_n$ *we have*

$$(\sum_i a_i)^2 \le n \sum_i a_i^2 \quad \textit{and} \quad \left(\frac{1}{n}\sum_i \frac{1}{a_i^2}\right)^{-1} \le \frac{1}{n}\sum_i a_i^2$$