

## Problem Set 3

You should hand in the solution for problem 4. Problems 1- 3 are for practice.

1. Let  $X_1, \dots, X_n$  be a random sample from a Poisson distribution with parameter  $\lambda$

$$P\{X = j\} = \frac{e^{-\lambda}\lambda^j}{j!} \quad j = 0, 1, \dots$$

- (a) Find the MLE of  $\lambda$  and its asymptotic distribution.
- (b) Assume that we are interested in estimating the probability of a count of zero  $\theta = P\{X = 0\} = \exp\{-\lambda\}$ . Find the MLE of  $\theta$  and its asymptotic distribution. *Hint:* you may use the delta-method.
- (c) Is the MLE of  $\theta$  you derived in (b) unbiased? Describe a bootstrap bias-correction you may do here.
- (d) Now consider a question of variance estimation. Find the asymptotic distribution of estimator  $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2$ . You may use the following facts about Poisson distribution:  $EX = \lambda, Var(X) = \lambda, E(X - EX)^3 = \lambda, E(X - EX)^4 = \lambda(1 + 3\lambda)$ .
- (e) Notice that since  $Var(X) = \lambda$ , the estimator in (a) is the MLE for variance in this model. How would you compare asymptotic efficiency of estimators in (a) and (d)?
2. Assume one observes random variables  $\{X_{i,t}, t = 1, 2, i = 1, \dots, n\}$  which are independent of each other and come from the following model:

$$X_{i,t} = \mu_i + \varepsilon_{i,t}, \text{ where } \varepsilon_{i,t} \sim N(0, \sigma^2)$$

The unknown parameter here is  $(\sigma^2, \mu_1, \mu_2, \dots, \mu_n)$ . This is the simplest panel data, and you can treat this situation as you observe each entity ( $i$ ) for two periods  $t = 1$  and  $t = 2$ , keeping in mind that each entity has its own unknown mean  $\mu_i$ . This is also known as fixed effects model.

- (a) Write down the likelihood function.
- (b) Find MLE for the unknown parameters.
- (c) Is the estimator for  $\mu_i$  unbiased? Consistent as  $n \rightarrow \infty$ ?
- (d) Is the MLE for  $\sigma^2$  unbiased? Consistent as  $n \rightarrow \infty$ ?
- (e) Why does asymptotic MLE theory fail to work in this case?

*The described problem is known as the incidental parameter problem and is extremely important for panel data analysis.*

3. Let  $X_1, \dots, X_n$  be a sample from the following distributions. In each case find the asymptotic variance of the MLE.

(a)  $f(x | \theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad 0 < \theta < \infty.$

(b)  $f(x | \theta) = \frac{1}{\theta} \exp\left\{-\frac{x}{\theta}\right\}, \quad 0 < x < \infty, \quad 0 < \theta < \infty.$

4. (*Required problem*) Suppose that income  $Y$  is distributed as a Pareto distribution:  $f(y) = \alpha y^{-(\alpha+1)}$  for  $1 \leq y$ , with  $\alpha > 1$ .

- (a) It is quite common to not observe all incomes, but only those that are higher than some threshold (so-called truncated variables). Assume that you observe only those individuals with an income greater than or equal to \$9,000, and their income is described by a random variable  $Y^*$ . How is  $Y^*$  distributed?
- (b) You have a sample of size  $N$  drawn from the population of persons with incomes greater than or equal to \$9,000. What is the MLE of  $\alpha$ ?
- (c) What is asymptotic distribution of the estimator in (b)?

- (d) Suppose you believe that the mean of the Pareto distribution out of which you draw an observation is affected linearly by a variable  $w$ , that is,  $E(Y_i|w_i) = \beta_1 + \beta_2 w_i$ . Assume that you have a sample of  $(Y_i^*, w_i)$  of size  $N$ . Explain how you would estimate the parameters  $\beta_1$  and  $\beta_2$ . *Hint: calculate the mean of the Pareto distribution. How is it related to  $\alpha$ ?*

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