14.451 Problem Set 4

Fall 2009

Due on October 26

1 Persistence and inertia

An agent can take two actions: $X = \{a, b\}$ and there are two shocks $Z = \{A, B\}$. The transition matrix between A and B is given by

$$\begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}.$$

The per-period payoffs are

$$F(x_t, x_{t+1}, z_t) = h(x_{t+1}, z_t) - l(x_t, x_{t+1})$$

where the function h is

$$\begin{aligned} h(a, A) &= h(b, B) = 1, \\ h(a, B) &= h(b, A) = 0, \end{aligned}$$

and the function l is

$$\begin{aligned} l(x, x') &= 0 \text{ if } x = x' \\ &= \xi \text{ if } x \neq x', \end{aligned}$$

where ξ is a non-negative scalar. In words: it is better to choose the action x_{t+1} appropriate to the current shock (a in A and b in B), but it is costly to change action.

1. Guess that the value function satisfies

   $$\overline{v} = V(a, A) = V(b, B) > V(b, A) = V(a, B) = \underline{v}$$

   and show that there are two regions in the space of the parameters (ξ, p): if the parameters lie in region 1 it is optimal to set $x_{t+1} = x_t$, if they are in region 2, it is optimal to set $x_{t+1} = a$ if $z_t = A$ and $x_{t+1} = b$ if $z_t = B$.

2. Show that increasing p (more persistent shocks) tend to make the agent more responsive (going from region 1 to region 2). Interpret.

3. Derive the optimal stochastic dynamics as a Markov chain on $S = X \times Z$.

4. Show that if the model parameters are in the interior of region 1, there are two ergodic sets. If they are in the interior of region 2, there is a unique ergodic set.

5. Derive the unique invariant distribution in the second case.

2 Invariant distributions and ergodic sets

Suppose you have a Markov chain which satisfies the property that for some j, $\pi_{i,j} > 0$ for all i. Let p^* be the unique invariant distribution.

1. Show that if E is an ergodic set then taking any p such that $\sum_{i \in E} p_i = 1$ we have $\sum_{i \in E} [pM]_i = 1$. In other words, Π maps probability distributions with $\sum_{i \in E} p_i = 1$ into probability distributions with the same property.

2. Prove that if E is an ergodic set and $i \notin E$ then $p_i^* = 0$. (Hint: use the contraction mapping theorem).

3. Prove that if E is an ergodic set then $p_i^* > 0$ for all $i \in E$. (Hint: go by contradiction, suppose there is an ergodic set which is a proper subset of E and show that then there exists an invariant distribution different from p^*.) Argue that the unique ergodic set is

$$E = \{i : p_i^* > 0\}.$$

3 Optimal control

Solve Excercises 7.4 and 7.13 in Acemoglu (2009).

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