14.451 Lecture Notes 5

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1 Example: optimal growth model

1.1 Characterizing the policy

We introduce the neoclassical optimal growth model and see how to derive properties of optimal investment and consumption from what we learned so far.

Identify x with the current capital stock, let

$$f(k) = F(k, 1) + (1 - \delta) k$$

which is output plus the undepreciated part of the capital stock (NB: the $F(k, 1)$ here is not the $F(x, y)$ in general notation, but it's ok because we'll use $f(k)$ from now on).

Let U be the utility function. We begin with an assumption that ensures consumers never throw away consumption goods:

Assumption 1. *The function $U(c)$ is increasing.*

Then they consume $f(k) - k'$ and the per-period payoff is

$$U(f(k) - k').$$

We will assume throughout that the following value function is well defined for all $k_0 > 0$:

$$V(k_0) \equiv \max \quad \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1})$$
$$0 \leq k_{t+1} \leq f(k_t), k_0 \text{ given.}$$

($V(0)$ may not be well defined as a max and $\lim_{k \to 0^+} V(k) = -\infty$ if $\lim_{c \to 0^+} U(c) = -\infty$).

We then can use the principle of optimality (in one direction) to argue that V must satisfy the Bellman equation

$$V(k) = \max_{0 \leq k' \leq f(k)} U(f(k) - k') + \beta V(k'). \qquad \text{(FE)}$$

Assumption 2. *The functions $f(k)$ and $U(c)$ are continuous, differentiable on the interior of their domain and strictly concave.*

The domain of f is always R_+. The domain of U can be R_+ or R_{++}. That is, in some cases we can have utility functions not defined for $c = 0$, with $U(c) \to -\infty$ as $c \to 0$ (e.g., when $U(c) = \log c$).

From concavity we have the following:

Claim 1 *$V(k)$ is strictly concave.*

Proof. This can be proved directly from the sequence problem. Take two initial conditions k_0 and k_0' and check that the sequence $\{\alpha k_t + (1-\alpha) k_t'\}$ is feasible and yields utility higher than the weighted average of $\{k_t\}$ and $\{k_t'\}$. ■

We now have some useful facts on concave functions (Fact 1 we have seen before but it's useful to remember).

Fact 1. If $h(x)$ is a concave function on a convex set $X \subset R^n$, then for each $x_0 \in X$ the function h has at least one subgradient p such that

$$h(x) - h(x_0) \leq p \cdot (x - x_0) \text{ for all } x \in X.$$

The set of all possible subgradients of h at x_0 is called the subdifferential of h and we denote with $\partial h(x_0)$.

Fact 2. If h is a function of one variable (i.e., X is a convex subset of R) then for each x the subgradient $\partial h(x)$ is a closed interval and is decreasing in x in the following sense:

$$\text{if } p \in \partial h(x) \text{ and } p' \in \partial h(x') \text{ then } p' \leq p \text{ if } x' > x,$$

and the mapping $\partial h(x)$ is upper hemicontinuous.

Fact 3. If h has a maximum at $x_0 \in X$ then $0 \in \partial h(x_0)$.

Fact 3 is a generalization of the fact that the derivative of a function differentiable at a local maximum is zero (and it includes the possibility of corner solutions).

But then using the last fact we can characterize the problem (FE) as follows: if the solution is interior we have

$$U'(f(k) - k') \in \beta \partial V(k'). \tag{1}$$

We now make an assumption to rule out the corner at zero consumption and to get a unique (and continuous) policy function $k' = g(k)$.

Assumption 3. *The utility function satisfies the Inada condition*

$$\lim_{c \to 0} U'(c) = \infty.$$

A graphical argument shows that there is a unique $k' < f(k)$ which satisfies (1). This is our policy function $g(k)$.

A graphical argument also shows the following:

Claim 2 *The policy $g(k)$ is continuous and non-decreasing in k. Optimal consumption $f(k) - g(k)$ is non-decreasing in k.*

We can now prove a Benveniste-Scheinkman type of result:

Claim 3 *The value function is differentiable for all $k > 0$ with*

$$V'(k) = U'(f(k) - g(k)) f'(k).$$

Proof. If $g(k) > 0$ we have an interior optimum and we can use B-S argument. Since $g(k)$ is non-decreasing we can only have $g(k) = 0$ on some interval $[0, k_1]$. On $(0, k_1)$ we have

$$V(k) = U(f(k)) + \beta V(0),$$

which is immediately differentiable. Moreover, we can show that the right and left derivative of V exist at k_1 and are equal (on the right use B-S argument, on the left our second argument!). So the function V is differentiable for all $k > 0$. ■

But then we can rule out the possibility of $g(k) = 0$ for any $k > 0$. For this we need one more assumption:

Assumption 4. *The production function satisfies $f(0) = 0$ (no output with no capital) and $\lim_{k \to 0^+} f'(k) > 0$ (something can be produced!).*

Take any $k > 0$. Notice that $f(k') - g(k') \to 0$ as $k' = 0$, using Assumption 3 there must exist some $\varepsilon > 0$ such that

$$U'(f(k) - \varepsilon) < \beta V'(\varepsilon) = U'(f(\varepsilon) - g(\varepsilon)) f'(\varepsilon).$$

This condition, together with concavity of U and V implies that there is a $g(k) > \varepsilon$ that satisfies the optimality condition

$$U'(f(k) - g(k)) = \beta V'(k') = U'(f(k') - g(k')) f'(k').$$

Since the optimum is unique $g(k) > 0$ for all $k > 0$. We have proved the following:

Claim 4 *The optimal policy is positive for all $k > 0$.*

1.2 Steady state and stability

First we look for a steady state. Then we will argue that there is a unique steady state with positive capital. Then that it is stable.

Definition: *A steady state is a k^* such that $k^* = g(k^*)$.*

There is always a steady state at $k^* = 0$, since then 0 is the only feasible continuation.

If there is a steady state with $k^* > 0$ it must satisfy

$$U'(f(k^*) - k^*) = \beta V'(k^*) = \beta U'(f(k^*) - k^*) f'(k^*).$$

This (and $U' > 0$) implies that

$$\beta f'(k^*) = 1. \tag{2}$$

Assumption 5. *The production function satisfies*

$$\lim_{k \to \infty} f'(k) = 0.$$

This implies that a solution to (2) exists.
Concavity of V implies that

$$\begin{aligned} V'(k) &> V'(g(k)) \iff k < g(k) \\ V'(k) &< V'(g(k)) \iff k > g(k) \end{aligned}$$

But if $k > 0$

$$\begin{aligned} V'(k) &= U'(f(k) - k) f'(k) \\ V'(g(k)) &= U'(f(k) - k) / \beta \end{aligned}$$

So we have

$$\begin{aligned} f'(k) &> 1/\beta \iff V'(k) > V'(g(k)) \\ f'(k) &< 1/\beta \iff V'(k) < V'(g(k)) \end{aligned}$$

and concavity of f implies

$$\begin{aligned} k &< k^* \iff f'(k) > 1/\beta \\ k &> k^* \iff f'(k) < 1/\beta \end{aligned}$$

So we conclude that if $k > 0$

$$\begin{aligned} k &< k^* \iff g(k) > k \\ k &> k^* \iff g(k) < k \end{aligned}$$

This shows that the steady state k^* is stable.

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