Damped Second-Order Systems

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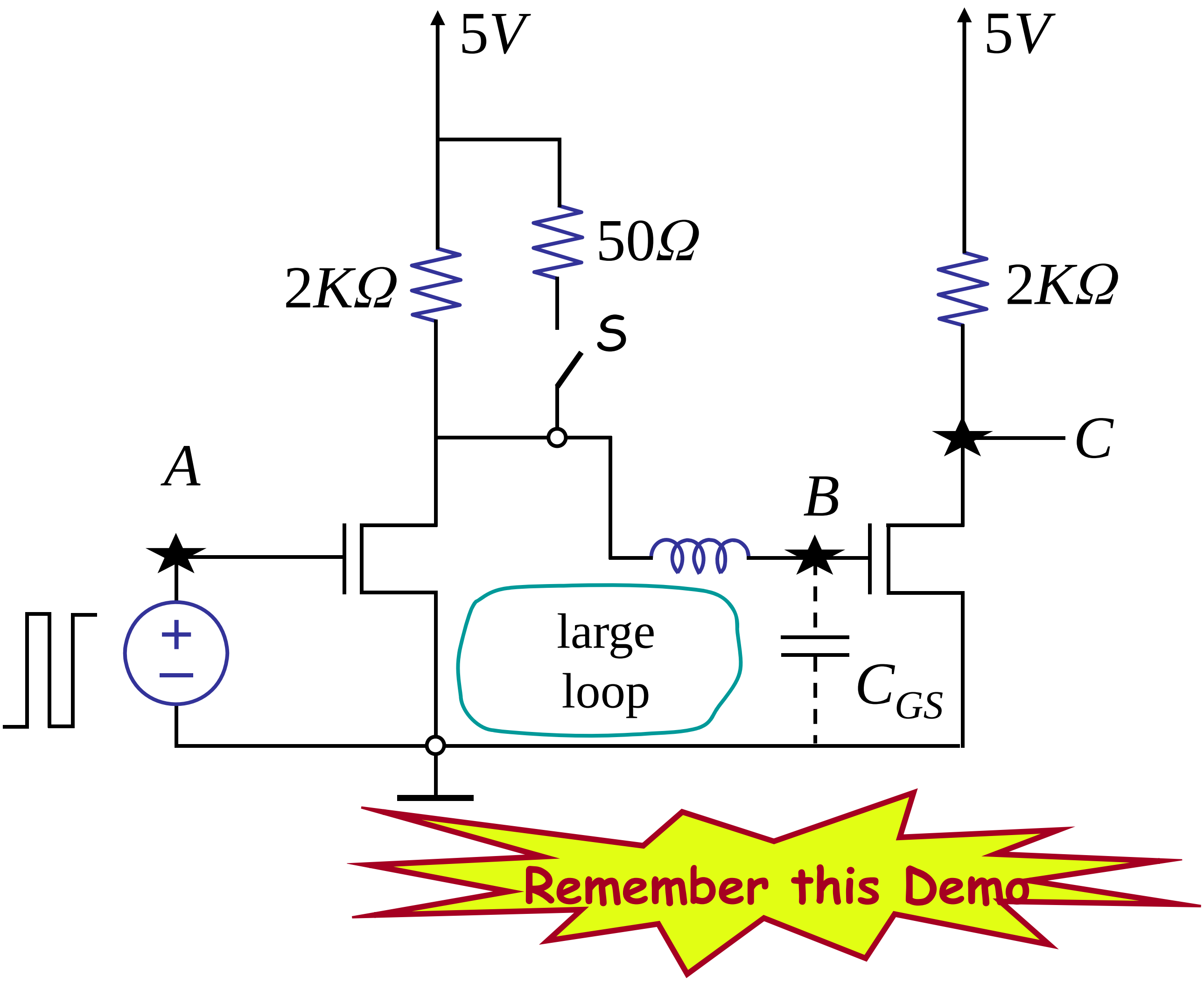
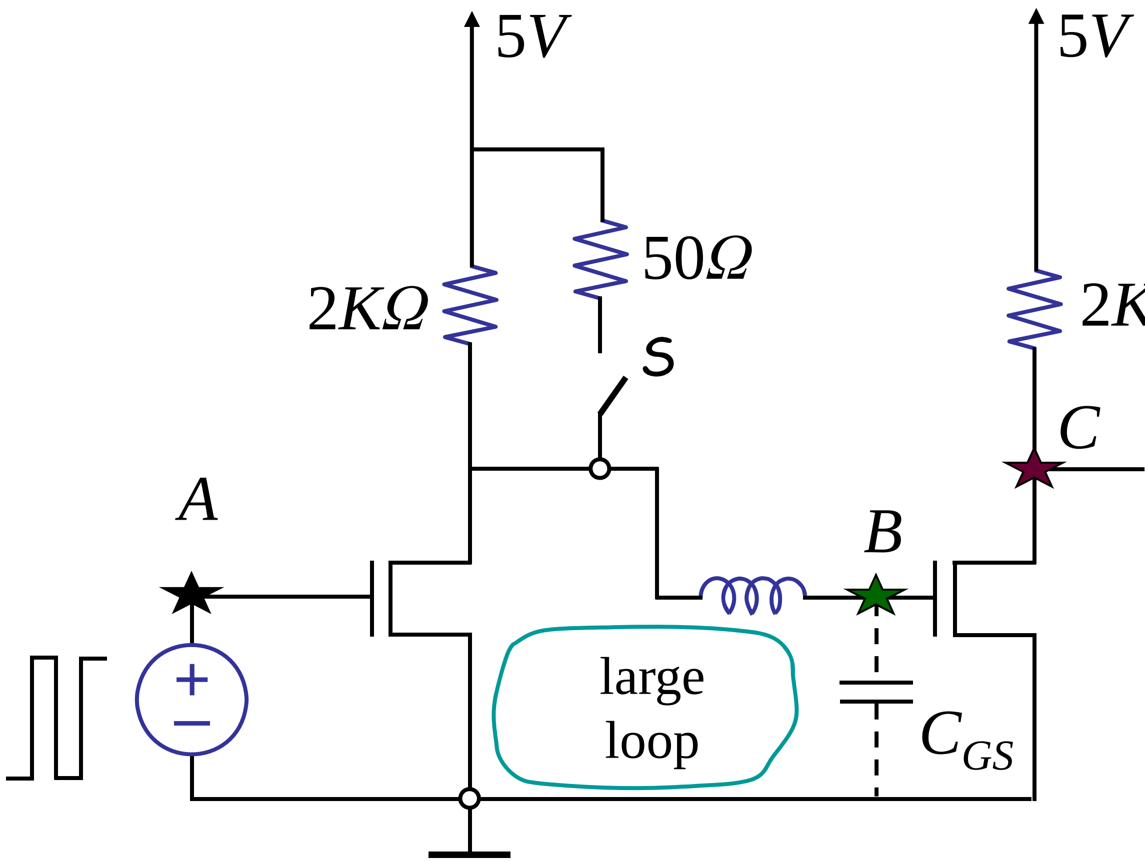


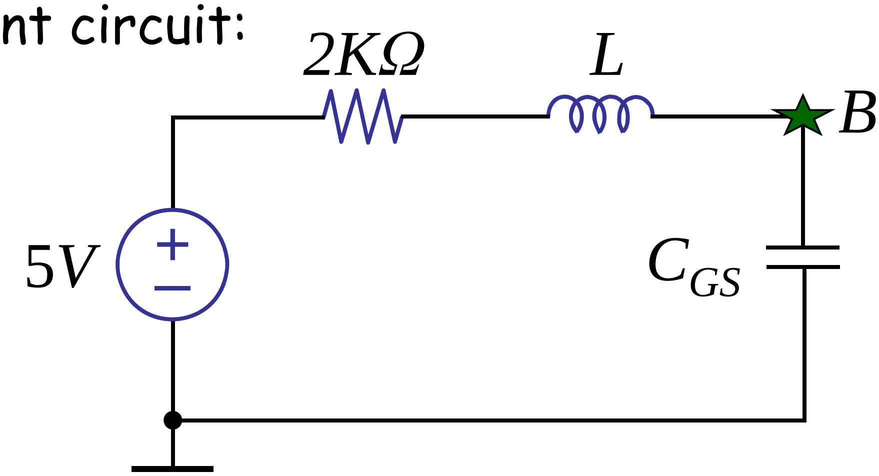
Our old friend, the inverter, driving another. The parasitic inductance of the wire and the gate-to-source capacitance of the MOSFET are shown

[Review complex algebra appendix in Agarwal & Lang for next class]

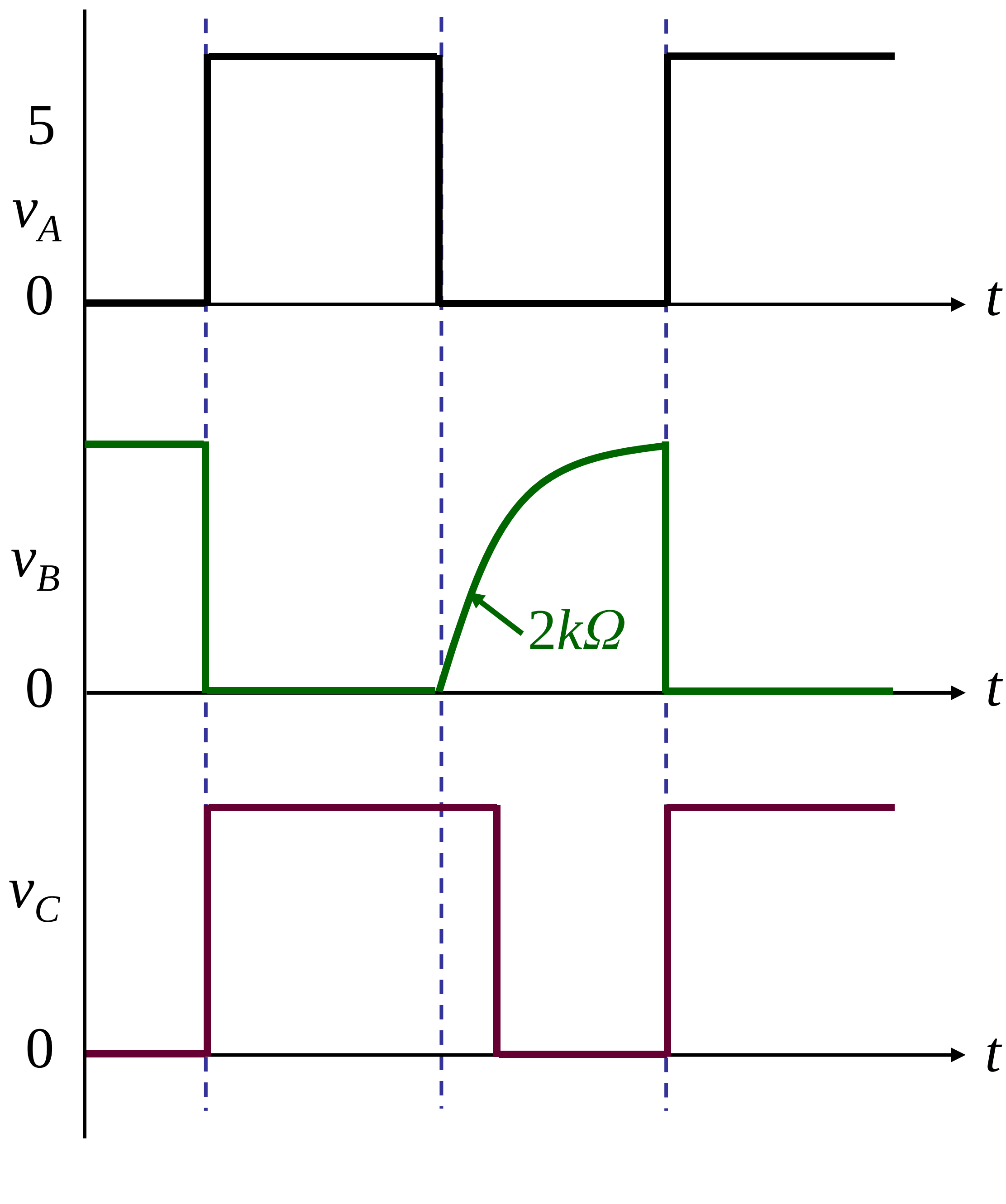
Damped Second-Order Systems



Relevant circuit:

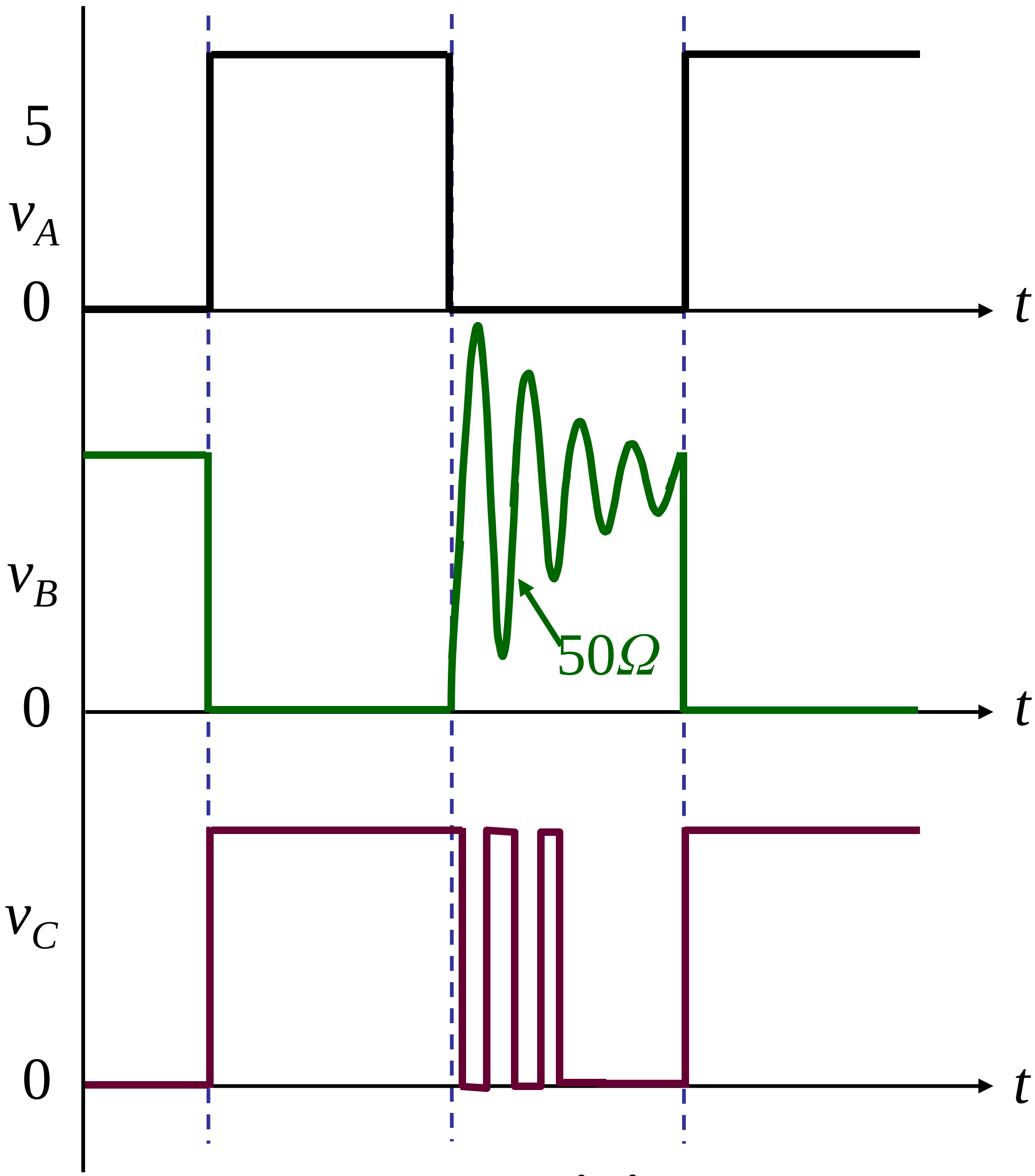


Observed Output $2k\Omega$



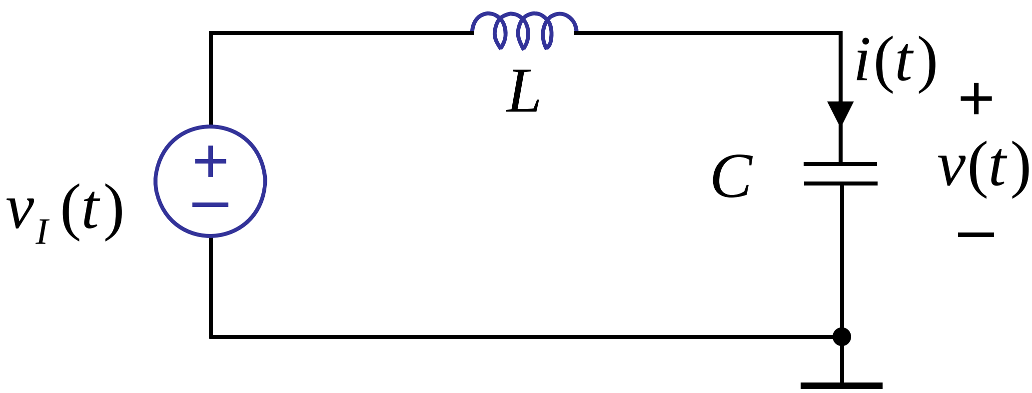
Now, let's try to speed up our inverter by closing the switch S to lower the effective resistance

Observed Output ~50Ω



Huh!

In the last lecture, we started by analyzing the simpler LC circuit to build intuition

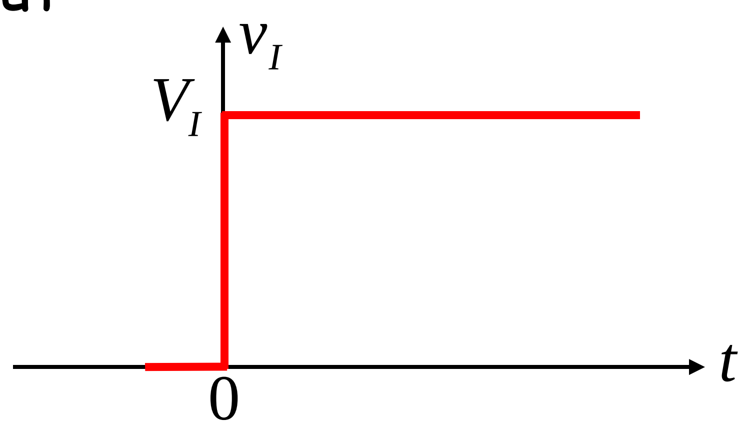


In the last lecture…

We solved

$$\frac{d^2v}{dt^2} + \frac{1}{LC}v = \frac{1}{LC}v_I$$

For input



And for initial conditions

$v(0) = 0 \qquad i(0) = 0 \qquad$ [ZSR]

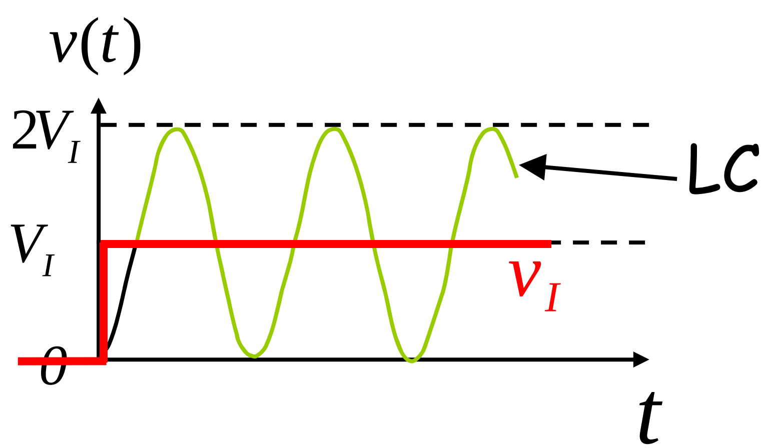
In the last lecture…

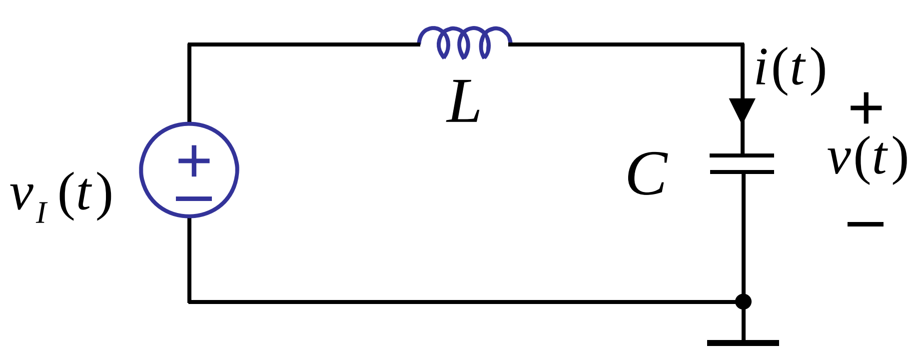
Total solution

$$v(t) = V_I - V_I \cos \omega_o t$$

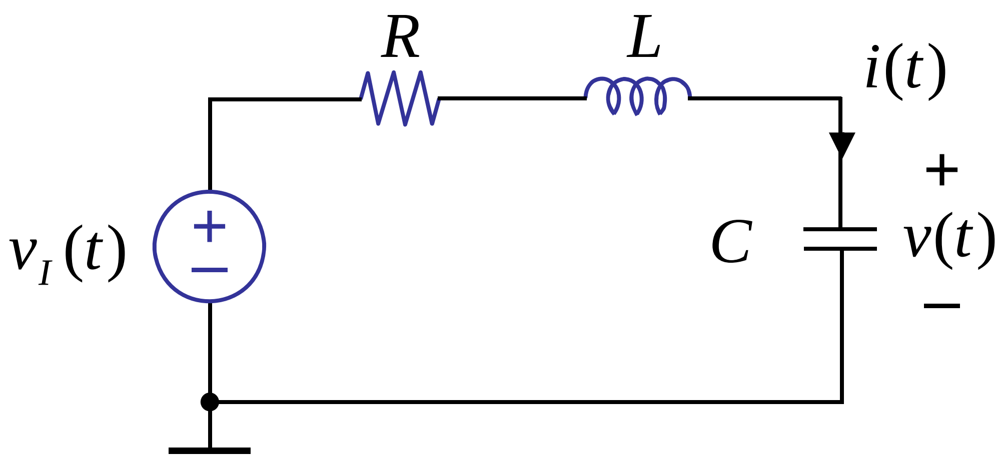
where

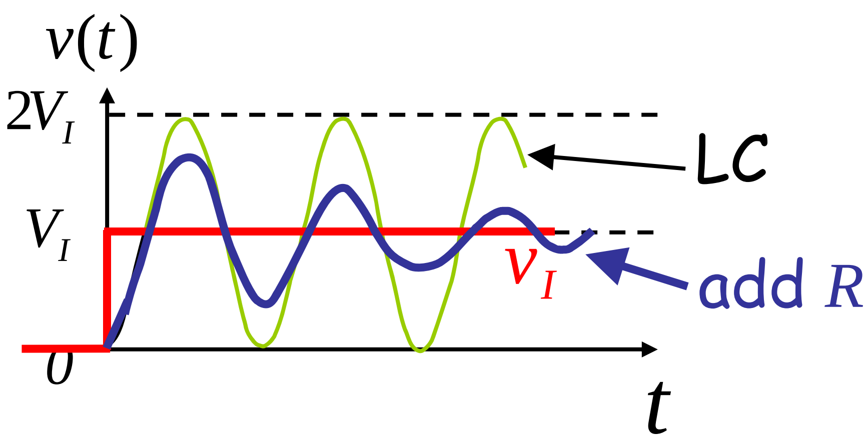
$$\omega_o = \frac{1}{\sqrt{LC}}$$





Today, we will close the loop on our observations in the demo by analyzing the RLC circuit

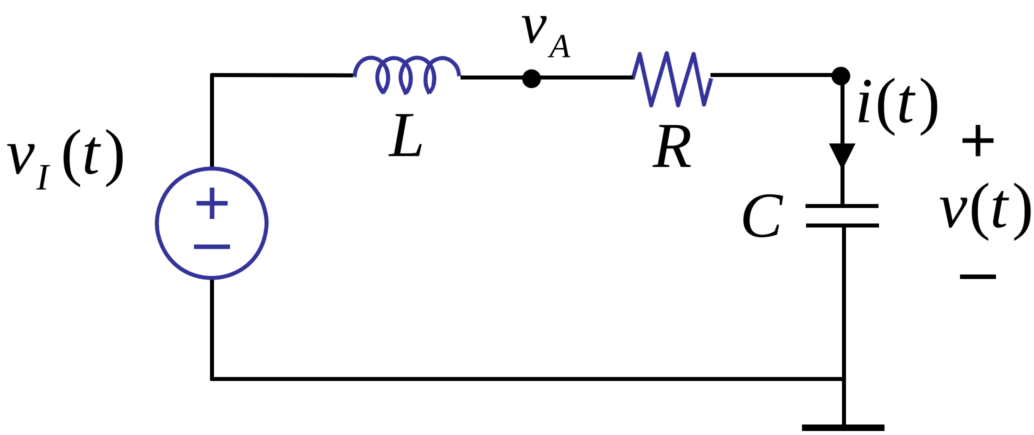




Damped sinusoids with R – remember demo!

See A&L Section 13.6

Let's analyze the RLC network



Node method:

$v_A:$ $$\frac{1}{L}\int_{-\infty}^{t}(v_I - v_A)\,dt = \frac{v_A - v}{R}$$

$v:$ $$\frac{v_A - v}{R} = C\frac{dv}{dt}$$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}v_I$$

Recall element rules

L:

$$v_L = L\frac{di}{dt}$$

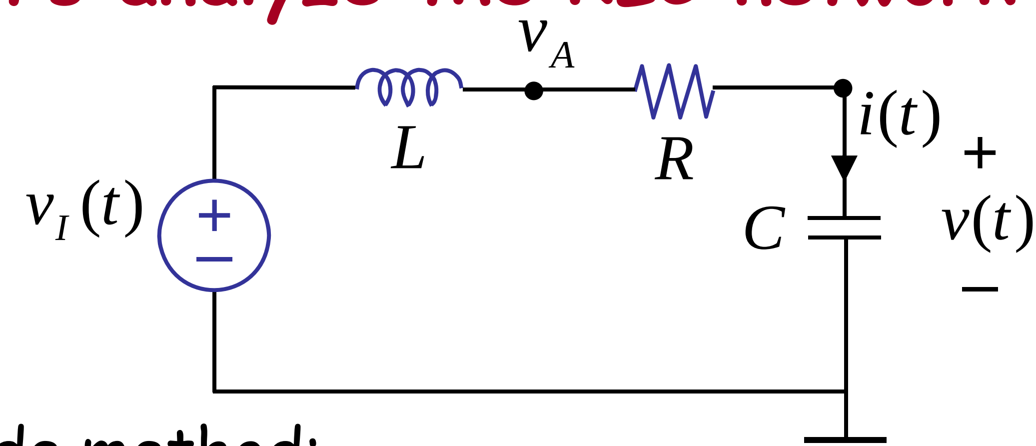
$$\frac{1}{L}\int_{-\infty}^{t} v_L\,dt = i$$

C:

$$i_C = C\frac{dv_C}{dt}$$

v, i state variables

Let's analyze the RLC network



Node method:

$$v_A: \quad \frac{1}{L}\int_{-\infty}^{t}(v_I - v_A)\,dt = \frac{v_A - v}{R}$$

$$v: \quad \frac{v_A - v}{R} = C\frac{dv}{dt}$$

$$\frac{1}{L}(v_I - v_A) = C\frac{d^2v}{dt^2}$$

$$\frac{1}{LC}(v_I - v_A) = \frac{d^2v}{dt^2}$$

$$v_A = RC\frac{dv}{dt} + v$$

$$\frac{1}{LC}\left(v_I - RC\frac{dv}{dt} - v\right) = \frac{d^2v}{dt^2}$$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}v_I$$

Solving

Recall, the method of homogeneous and particular solutions:

① Find the particular solution.

② Find the homogeneous solution.
↓
4 steps

③ The total solution is the sum of the particular and homogeneous.

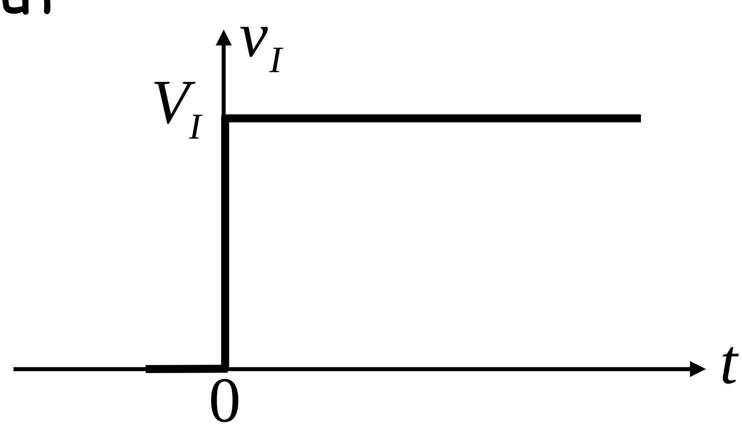
Use initial conditions to solve for the remaining constants.

$$v = v_P(t) + v_H(t)$$

Let's solve

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}v_I$$

For input



And for initial conditions

$v(0) = 0 \qquad i(0) = 0 \qquad$ [ZSR]

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① Particular solution

$$\frac{d^2 v_P}{dt^2} + \frac{R}{L}\frac{dv_P}{dt} + \frac{1}{LC} v_P = \frac{1}{LC} V_I$$

$v_P = V_I$ is a solution.

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② Homogeneous solution

Solution to $$\frac{d^2 v_H}{dt^2} + \frac{R}{L}\frac{dv_H}{dt} + \frac{1}{LC} v_H = 0$$

Recall, v_H : solution to homogeneous equation (drive set to zero)

Four-step method:

Ⓐ Assume solution of the form

$$v_H = Ae^{st} \qquad , \; A, \; s \; = \; ?$$

Ⓑ Form the characteristic equation f(s)

Ⓒ Find the roots of the characteristic equation

$$s_1, s_2$$

Ⓓ General solution

$$v_H = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

② Homogeneous solution

Solution to $$\frac{d^2 v_H}{dt^2} + \frac{R}{L}\frac{dv_H}{dt} + \frac{1}{LC} v_H = 0$$

Ⓐ Assume solution of the form

$$v_H = Ae^{st} \qquad , \; A, \; s \; = \; ?$$

so, $$As^2 e^{st} + \frac{R}{L} Ase^{st} + \frac{1}{LC} Ae^{st} = 0$$

Ⓑ $$s^2 + \frac{R}{L} s + \frac{1}{LC} = 0$$ characteristic equation

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

$$\omega_o = \sqrt{\frac{1}{LC}}$$

$$\alpha = \frac{R}{2L}$$

Ⓒ Roots $$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2}$$
$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

Ⓓ General solution

$$v_H = A_1 e^{\left(-\alpha + \sqrt{\alpha^2 - \omega_o^2}\right)t} + A_2 e^{\left(-\alpha - \sqrt{\alpha^2 - \omega_o^2}\right)t}$$

③ Total solution

$$v(t) = v_P(t) + v_H(t)$$

$$v(t) = V_I + A_1 e^{\left(-\alpha + \sqrt{\alpha^2 - \omega_o^2}\right)t} + A_2 e^{\left(-\alpha - \sqrt{\alpha^2 - \omega_o^2}\right)t}$$

Find unknowns from initial conditions.

$v(0) = 0: \quad 0 = V_I + A_1 + A_2$

$i(0) = 0:$

$$i(t) = C\frac{dv}{dt}$$

$$= CA_1\left(-\alpha + \sqrt{\alpha^2 - \omega_o^2}\right)e^{\left(-\alpha + \sqrt{\alpha^2 - \omega_o^2}\right)t} +$$

$$CA_2\left(-\alpha - \sqrt{\alpha^2 - \omega_o^2}\right)e^{\left(-\alpha - \sqrt{\alpha^2 - \omega_o^2}\right)t}$$

so, $\quad 0 = A_1\left(-\alpha + \sqrt{\alpha^2 - \omega_o^2}\right) + A_2\left(-\alpha - \sqrt{\alpha^2 - \omega_o^2}\right)$

Mathematically: solve for unknowns, done.

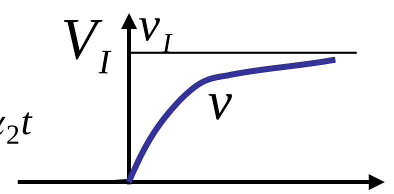
Let's stare at this a while longer…

$$v(t) = V_I + A_1 e^{-\alpha t} e^{\left(\sqrt{\alpha^2 - \omega_o^2}\right)t} + A_2 e^{-\alpha t} e^{\left(-\sqrt{\alpha^2 - \omega_o^2}\right)t}$$

3 cases:

$\alpha > \omega_o$ Overdamped

$$v(t) = V_I + A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t}$$



$\alpha < \omega_o$ Underdamped

$$v(t) = V_I + A_1 e^{-\alpha t} e^{\left(j\sqrt{\omega_o^2 - \alpha^2}\right)t} + A_2 e^{-\alpha t} e^{\left(-j\sqrt{\omega_o^2 - \alpha^2}\right)t}$$

$$= V_I + A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t}$$

$$= V_I + K_1 e^{-\alpha t} \cos\omega_d t + K_2 e^{-\alpha t} \sin\omega_d t$$

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$$

$$e^{j\omega_d t} = \cos\omega_d t + j\sin\omega_d t$$

$\alpha = \omega_o$ Critically damped

Later…

Let's stare at underdamped a while longer…

$\alpha < \omega_o$ Underdamped contd…

$$v(t) = V_I + K_1 e^{-\alpha t}\cos\omega_d t + K_2 e^{-\alpha t}\sin\omega_d t$$

$$v(0) = 0: \quad K_1 = -V_I$$

$$i(0) = 0: \; i(t) = C\frac{dv}{dt}$$

$$= -CK_1 \alpha e^{-\alpha t}\cos\omega_d t - CK_2\omega_d e^{-\alpha t}\sin\omega_d t$$

$$- CK_1 \alpha e^{-\alpha t}\sin\omega_d t + CK_2\omega_d e^{-\alpha t}\cos\omega_d t$$

$$0 = -K_1\alpha + K_2\omega_d$$

$$K_2 = -\frac{V_1\alpha}{\omega_d}$$

$$v(t) = V_I - V_I e^{-\alpha t}\cos\omega_d t - V_I \frac{\alpha}{\omega_d} e^{-\alpha t}\sin\omega_d t$$

Note: For $R = 0 \Longrightarrow \alpha = 0$

$$v(t) = V_I - V_I \cos\omega_o t$$

Same as LC as expected

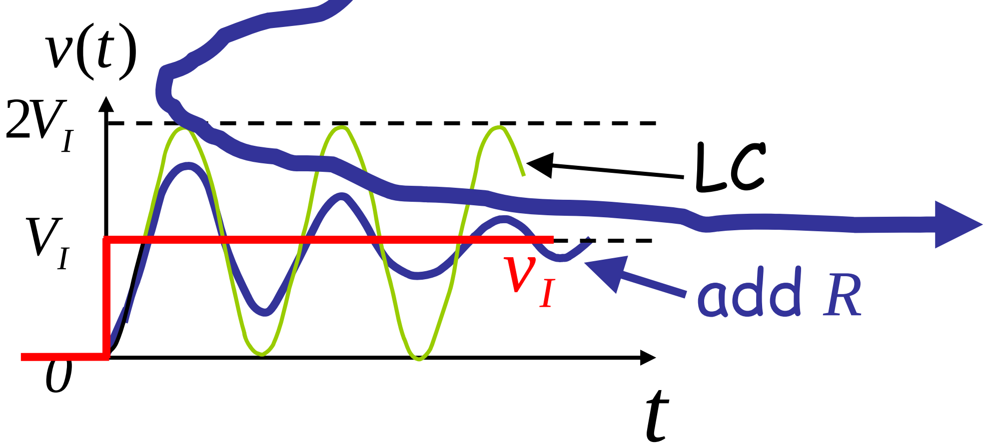
Let's stare at underdamped a while longer…

$\alpha < \omega_o$ Underdamped contd…

$$v(t) = V_I - V_I e^{-\alpha t}\cos\omega_d t - V_I \frac{\alpha}{\omega_d} e^{-\alpha t}\sin\omega_d t$$

Remember, scaled sum of sines (of the same frequency) are also sines! -- Appendix B.7

$$v(t) = V_I - V_I \frac{\omega_o}{\omega_d} e^{-\alpha t}\cos\left(\omega_d t - \tan^{-1}\frac{\alpha}{\omega_d}\right)$$

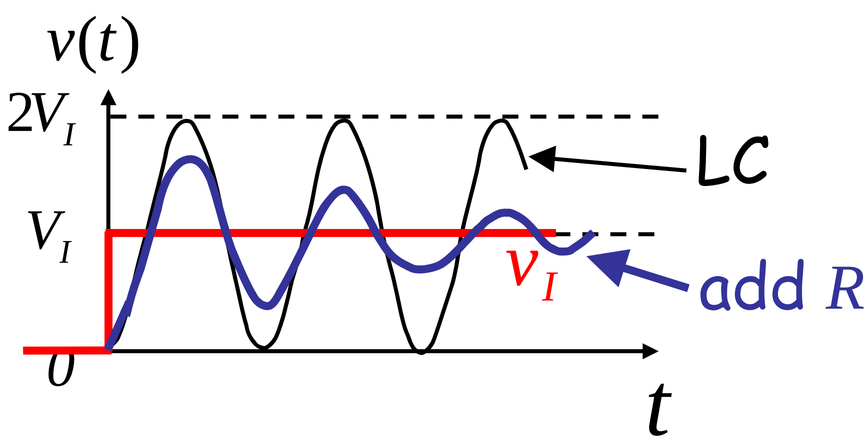


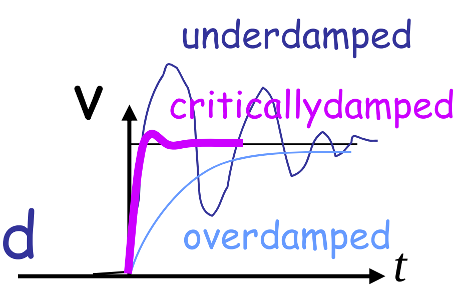
$\alpha < \omega_o$ Underdamped contd…

$$v(t) = V_I - V_I e^{-\alpha t}\cos\omega_d t - V_I \frac{\alpha}{\omega_d} e^{-\alpha t}\sin\omega_d t$$

Remember, scaled sum of sines (of the same frequency) are also sines! -- Appendix B.7

$$v(t) = V_I - V_I \frac{\omega_o}{\omega_d} e^{-\alpha t}\cos\left(\omega_d t - \tan^{-1}\frac{\alpha}{\omega_d}\right)$$

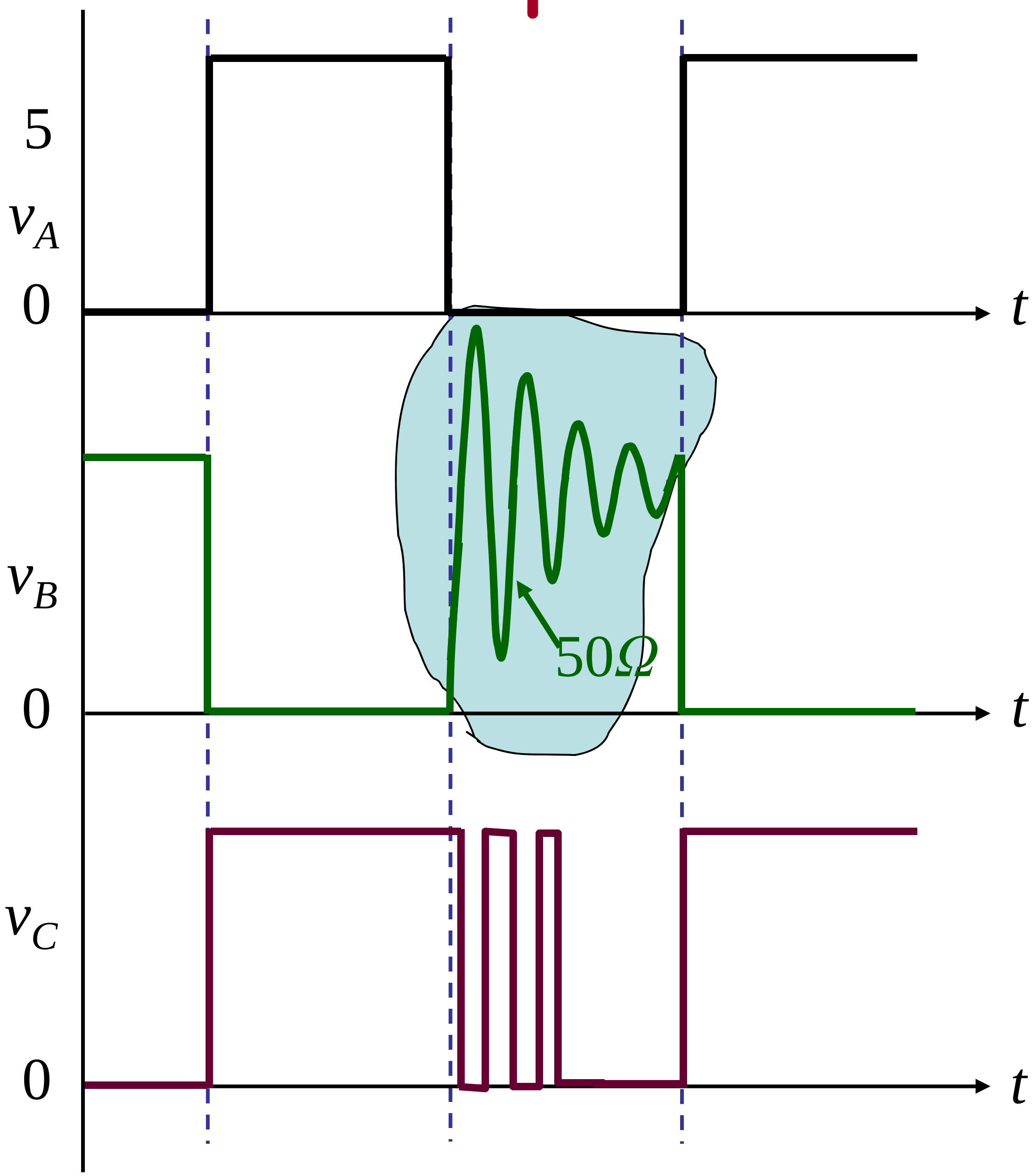




$\alpha = \omega_o$ Critically damped

Section 13.2.3

Remember this? Closed the loop…

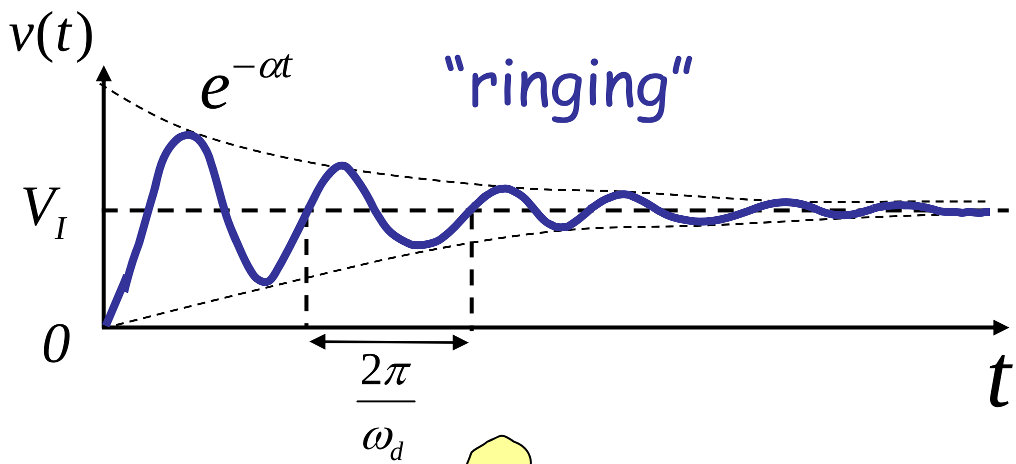


See example 12.9 on page 664 of the A&L textbook for inverter-pair analysis

Intuitive Analysis

See Sec. 12.7 of A&L textbook

Underdamped $$v(t) = V_I - V_I \frac{\omega_o}{\omega_d} e^{-\alpha t} \cos\left(\omega_d t - \tan^{-1}\frac{\alpha}{\omega_d}\right)$$



Characteristic equation

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

ω_d : Oscillation frequency $\omega_d = \sqrt{\omega_o^2 - \alpha^2}$

α : Governs rate of decay

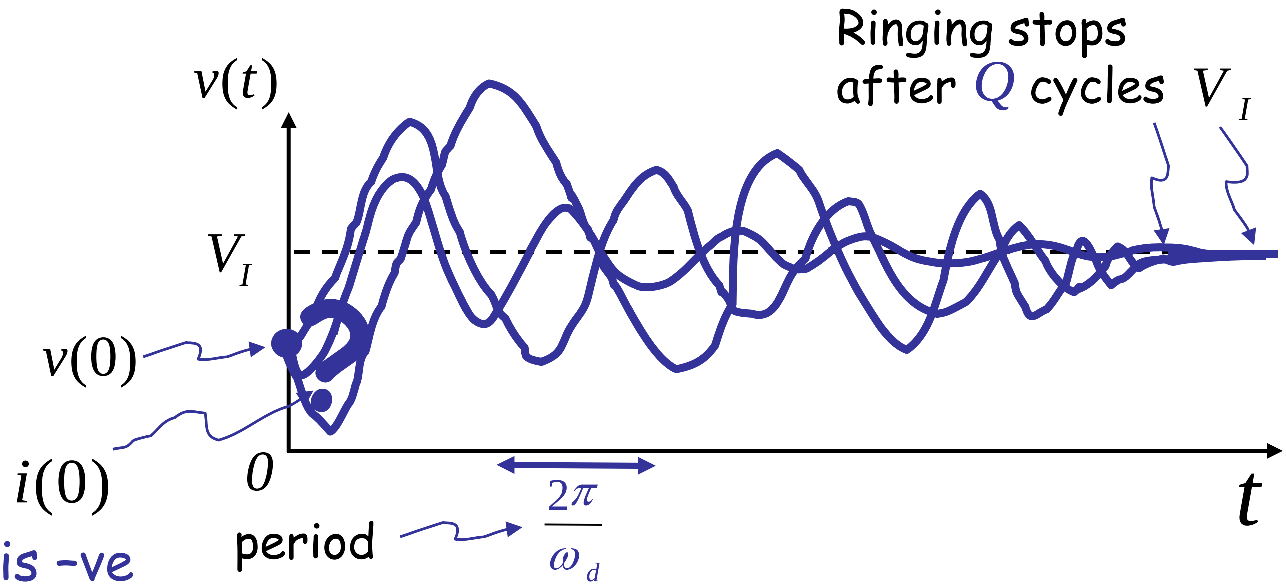
V_I : Final value

$v(0)$: Initial value

$Q = \frac{\omega_o}{2\alpha}$: Quality factor (approximately the number of cycles of ringing)

Intuitive Analysis

See Sec. 12.7
of A&L textbook



is -ve
so v(t)
must
drop

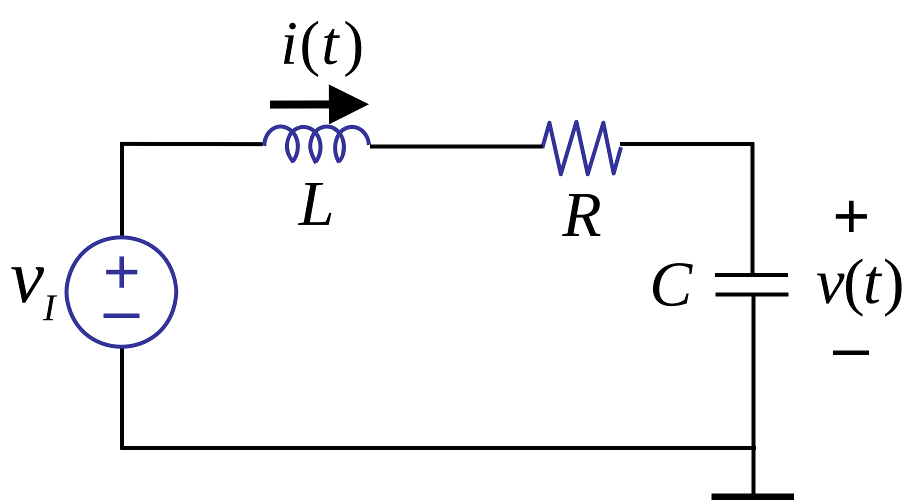
Characteristic
equation

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s^2 + 2\alpha s + \omega_o^2 = 0$$

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$$

$$Q = \frac{\omega_o}{2\alpha}$$



given
$i(0)$ -ve
$v(0)$ +ve