Electromagnetic Waves

Reading – Shen and Kong – Ch. 3

Outline

Review of the Quasi-static Approximation
Electric and Magnetic Components of Waves
The Wave Equation (in 1-D)
Uniform Plane Waves
Phase Velocity and Intrinsic Impedance
Wave-vector and Wave-frequency

Maxwell's Equations (Free Space with Charges)

	Differential form	**Integral form**
E-Gauss:	$\nabla \cdot \epsilon_o \vec{E} = \rho$	$\oiint_S \epsilon_o \vec{E} \cdot dS = \iiint_V \rho dV$
Faraday:	$\nabla \times \vec{E} = -\frac{\partial}{\partial t} \mu_o \vec{H}$	$\oint_C \vec{E} \cdot d\vec{C} = \iint_S -\frac{\partial}{\partial t} \mu_o \vec{H} \cdot d\vec{S}$
H-Gauss:	$\nabla \cdot \mu_o \vec{H} = 0$	$\oiint_S \mu_o \vec{H} \cdot d\vec{S} = 0$
Ampere:	$\nabla \times \vec{H} = \vec{J} + \frac{\partial}{\partial t} \epsilon_o \vec{E}$	$\oint_C \vec{H} \cdot d\vec{C} = \iint_S (\vec{J} + \frac{\partial}{\partial t} \mu_o \vec{H}) \cdot d\vec{S}$

In statics, both time derivatives are unimportant, Maxwell's Equations split into decoupled electrostatic and magnetostatic equations.
In Electro-quasistatic (EQS) and magneto-quasitatic systems (MQS), one (but not both) time derivative becomes important.

Quasi-static Maxwell's Equations

Electric Fields

$$\oint_S \epsilon_o \vec{E} \cdot d\vec{A} = \int_V \rho dV$$

EQS

$$\oint_C \vec{E} \cdot d\vec{l} = 0$$

$$\frac{E_{error}}{E} = \omega^2 \mu \epsilon L^2$$

Magnetic Fields

$$\oint_S \vec{B} \cdot d\vec{A} = 0$$

MQS

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{A}$$

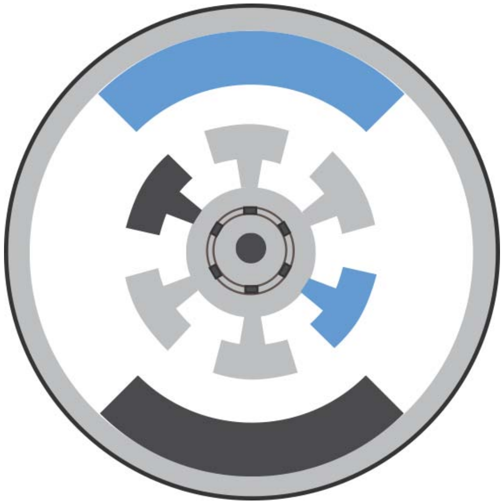
$$\frac{H_{error}}{H} = \omega^2 \mu \epsilon L^2$$

For the error in the QS approximation to be small … $\omega L \ll \frac{1}{\sqrt{\mu\epsilon}}$ or $L \ll \frac{\lambda}{2\pi}$

EQS vs MQS for Time-Varying Fields

Why did we not worry about the magnetic field generated by the time-varying electric field of a motor ?

$$\omega L \ll \frac{1}{\sqrt{\mu\epsilon}}$$



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$$\left.\begin{array}{l} \omega = 2\pi \ 2000 \text{ rpm} \\ L = 0.01 \text{ m} \end{array}\right\} \omega L = 120 \ \frac{\text{m}}{\text{s}} \ll \frac{1}{\sqrt{\mu_o \epsilon_o}} \approx 3\text{x}10^8 \frac{\text{m}}{\text{s}}$$

A typical motor frequency of 2000 rpm satisfies EQS approximation for free-space

<u>As another example, note</u>:
At 60 Hz, the wavelength (typical length) in air is 5000 km, therefore, almost all physical 60-Hz systems in air are quasistatic (since they are typically smaller than 5000 km in size)

Coupling of Electric and Magnetic Fields

Maxwell's Equations couple the E and H fields:

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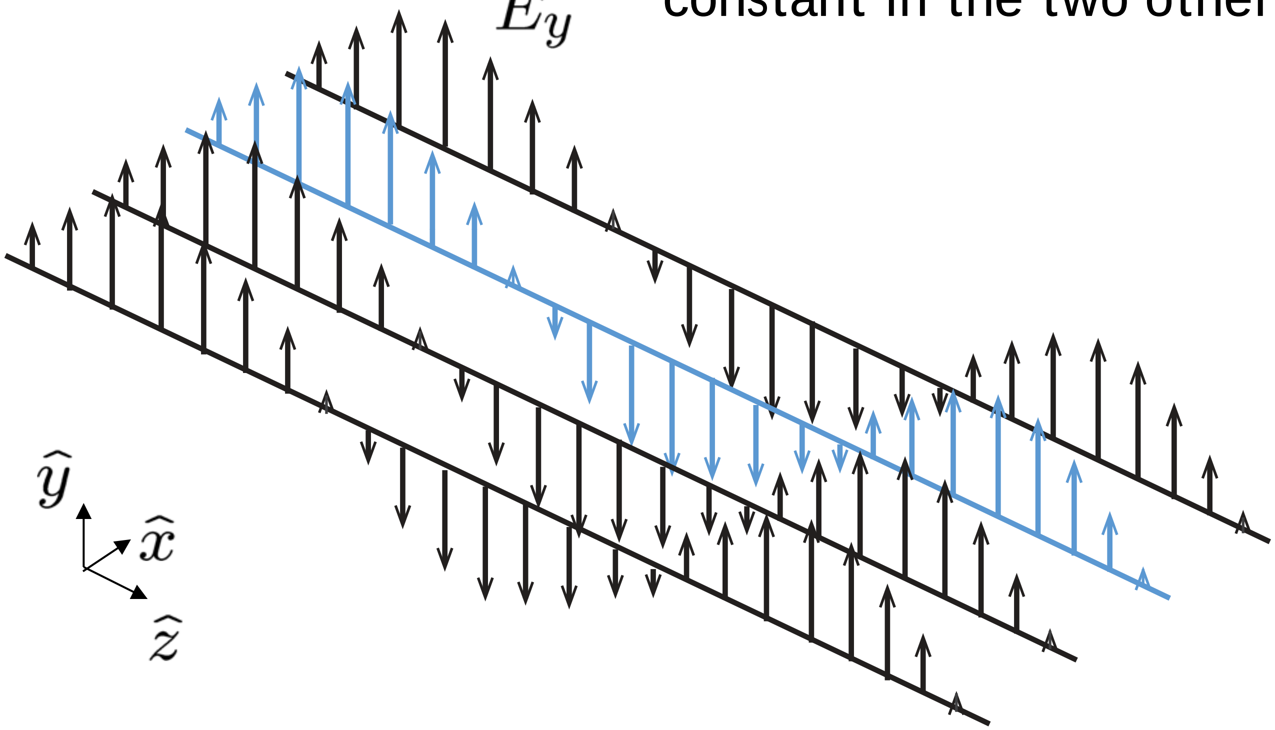
$$\oint_C \overline{E} \cdot d\bar{l} = -\frac{d}{dt}\left(\int_S \overline{B} \cdot d\overline{A}\right)$$

$$\oint_C \overline{H} \cdot d\bar{l} = \int_S \overline{J} \cdot d\overline{A} + \frac{d}{dt}\int_S \epsilon E dA$$

Uniform Electromagnetic Waves

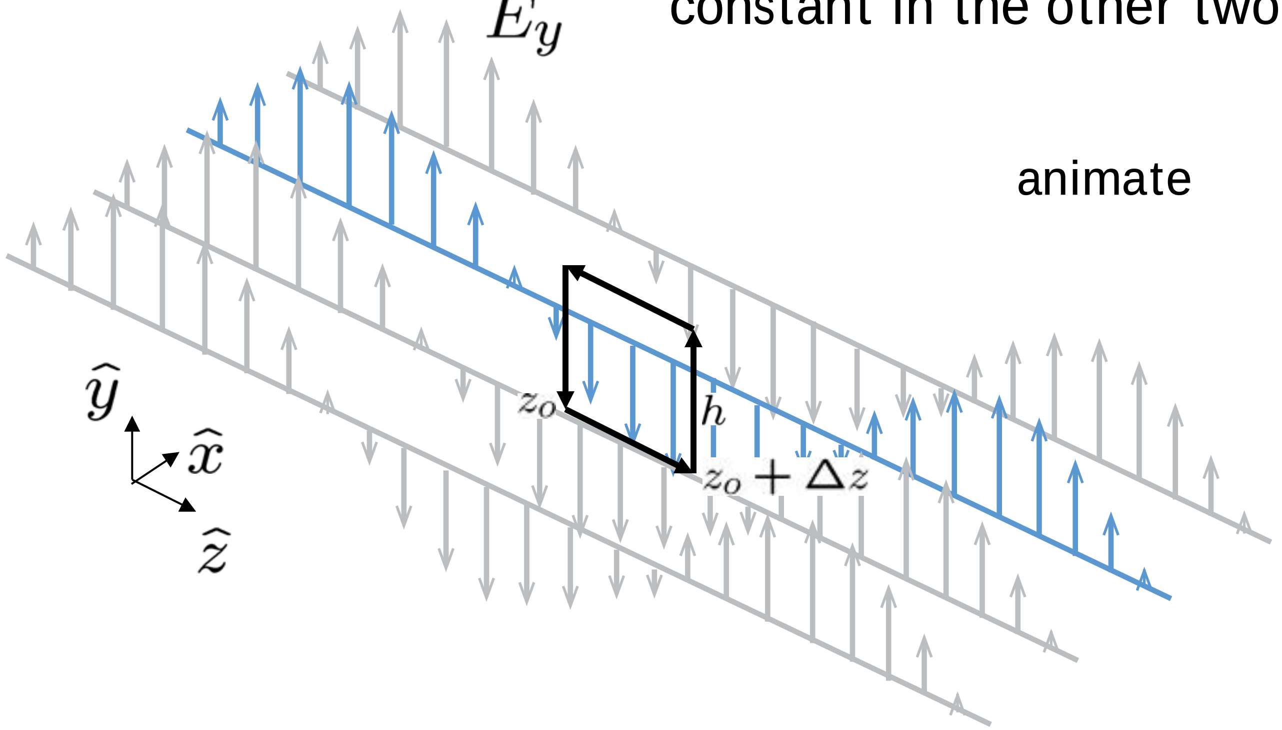
E_y varies along the z-direction and E is constant in the two other directions



$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\left(\int_S \vec{B} \cdot d\vec{A}\right)$$

Uniform Electromagnetic Waves

E_y varies along the z-direction and E_y is constant in the other two directions



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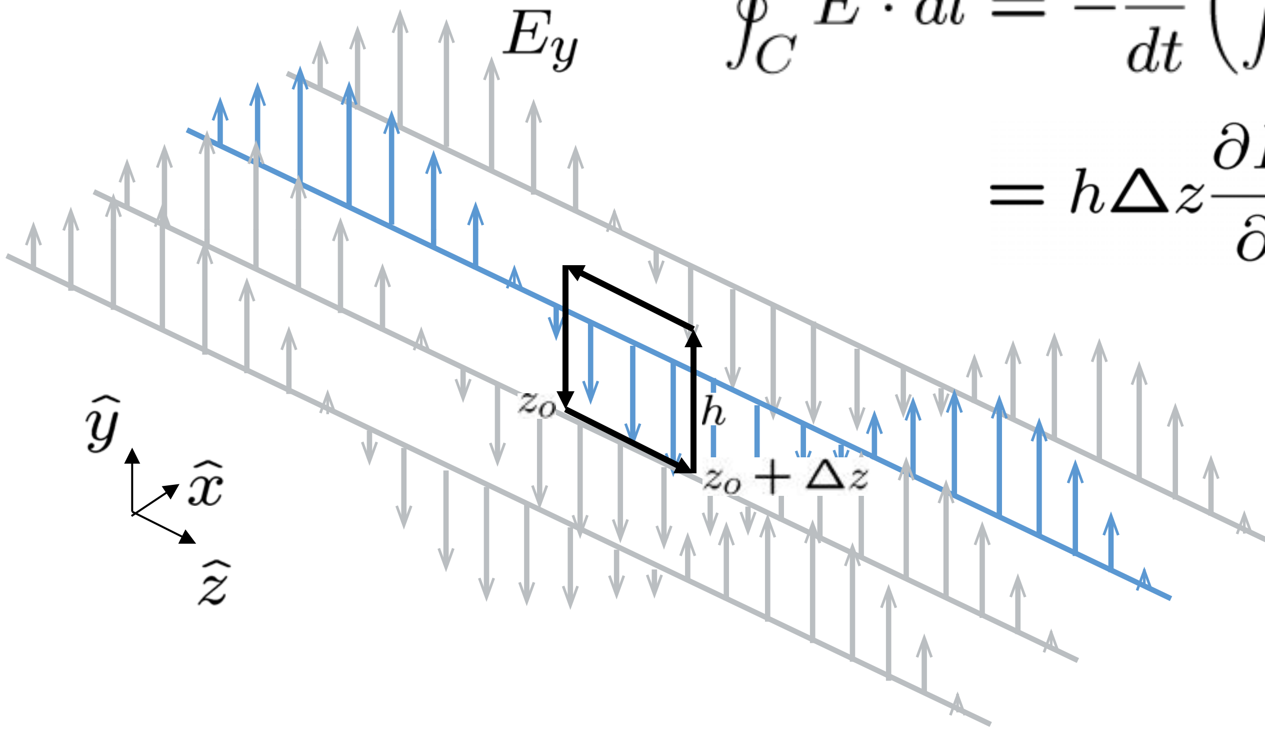
$$\oint_C \vec{E} \cdot d\vec{l} = \quad hE_y(z_o + \Delta z) - hE_y(z_o) \quad = -\frac{d}{dt}\left(\int_S \vec{B} \cdot d\vec{A}\right)$$

$$= h\Delta z \, \frac{E_y(z_o + \Delta z) - E_y(z_o)}{\Delta z} \; = h\Delta z \frac{\partial E_y}{\partial z}$$

Electromagnetic Waves



$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\left(\int_S \vec{B} \cdot d\vec{A}\right)$$

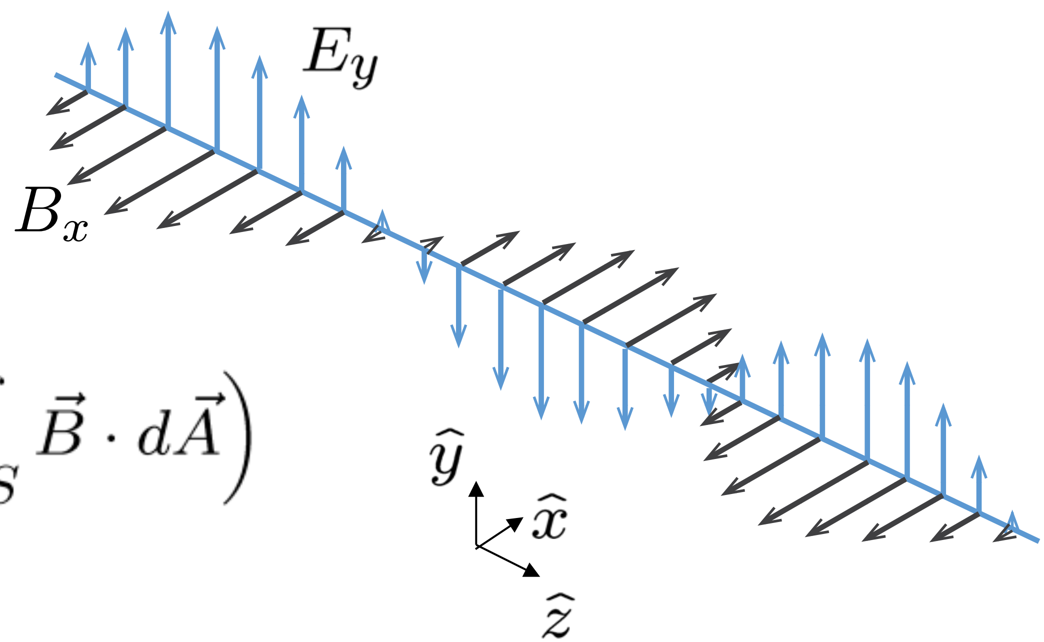
$$= h\Delta z \frac{\partial E_y}{\partial z}$$

E_y-field cannot vary in z-direction without a time-varying B-field ...

$$-\frac{\partial}{\partial t}\int \vec{B} d\vec{A} = -\frac{\partial}{\partial t}(-B_x h \Delta z) = h\Delta z \frac{\partial B_x}{\partial t}$$

...and waves must have both electric and magnetic components !

Uniform Electromagnetic Plane Waves

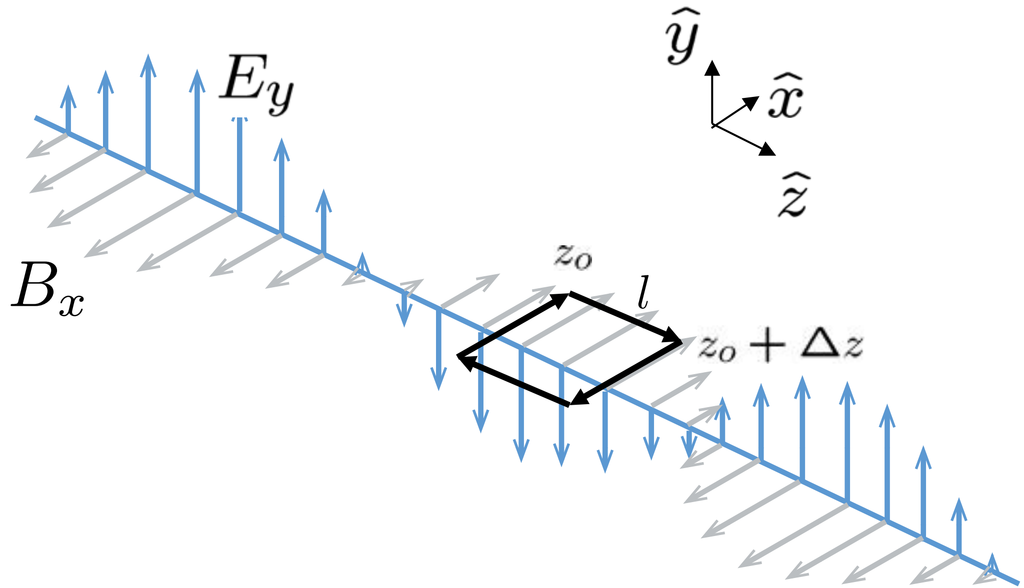


$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\left(\int_S \vec{B} \cdot d\vec{A}\right)$$

$$\frac{\partial E_y}{\partial z} = \frac{\partial B_x}{\partial t}$$

The **y-component of E** that varies across space is associated with the **x-component of B** that varies in time

Uniform Electromagnetic Plane Waves



$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{A} + \frac{d}{dt} \int_S \epsilon_o \vec{E} \, d\vec{A}$$

$$\frac{\partial B_x(z_o)}{\partial z} = \epsilon_o \mu_o \frac{\partial E_y}{\partial t}$$

Source free: $J = 0$

The Wave Equation

Time-varying E_y generates spatially varying B_z ...

$$\frac{\partial^2 B_x(z_o)}{\partial z\,\partial t} = \epsilon_o \mu_o \frac{\partial^2 E_y}{\partial t^2}$$

Time-varying B_z generates spatially varying E_y ...

$$\frac{\partial^2 E_y}{\partial z^2} = \frac{\partial^2 B_x(z_o)}{\partial t\,\partial z}$$

The temporal and spatial variations in E_y are coupled together to yield ...

$$\boxed{\frac{\partial^2 E_y}{\partial z^2} = \epsilon_o \mu_o \frac{\partial^2 E_y}{\partial t^2}}$$

... the Wave Equation.

The Wave Equation via Differential Equations

Faraday: $$\underbrace{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}}_{\nabla \times \vec{E}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix} = -\hat{x}\frac{\partial E_y}{\partial z} = -\hat{x}\frac{\partial \mu H_x}{\partial t}$$

Ampere: $$\underbrace{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}}_{\nabla \times \vec{H}} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & 0 & 0 \end{vmatrix} = -\hat{y}\frac{\partial H_x}{\partial z} = -\hat{y}\frac{\partial \epsilon E_y}{\partial t}$$

Substitution yields the wave equation: $$\frac{\partial^2 E_y}{\partial z^2} = \epsilon_o \mu_o \frac{\partial^2 E_y}{\partial t^2}$$

Uniform Plane Wave Solutions

The 1-D wave equation

$$\frac{\partial^2 E_y}{\partial z^2} - \epsilon_o \mu_o \frac{\partial^2 E_y}{\partial t^2} = 0$$

- $E_y(z,t)$ is any function for which the second derivative in space equals its second derivative in time, times a constant. The solution is therefore any function with the same dependence on time as on space, e.g.

$$E_y = f_+(t - z/c) + f_-(t + z/c)$$

- The functions ***f_+(z-ct)*** and ***f_- (z+ct)*** represent uniform waves propagating in the +*z* and -*z* directions respectively.

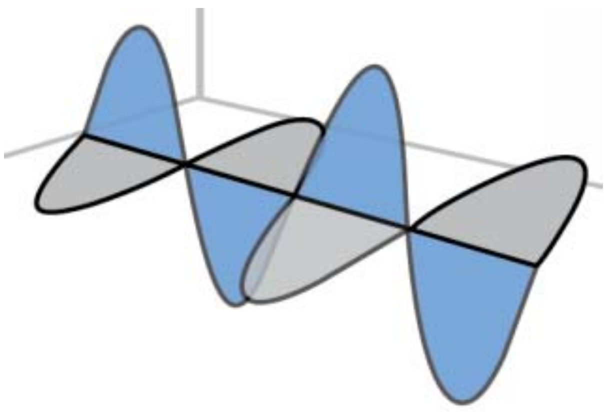
Speed of Light

- The ***velocity of propagation*** is determined solely by the dielectric permittivity and magnetic permeability:

$$c = \frac{1}{\sqrt{\epsilon_o \mu_o}}$$

- The functions f_+ and f_- are determined by the source and the other boundary conditions.

$$E_y = f_+(t - z/c) + f_-(t + z/c)$$



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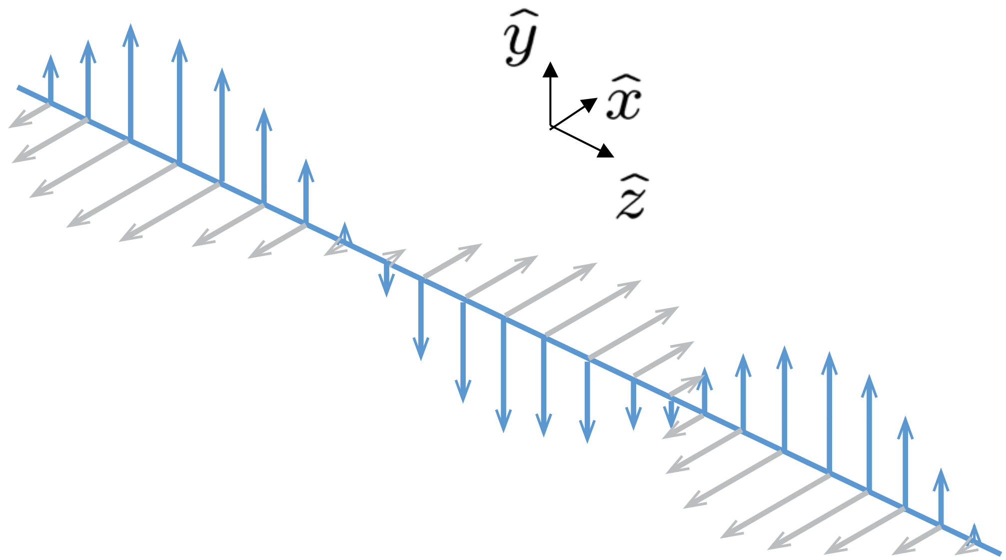
Magnetic Field of a Uniform Plane Wave

$$\frac{\partial B_x(z_o)}{\partial z} = \epsilon_o \mu_o \frac{\partial E_y}{\partial t}$$



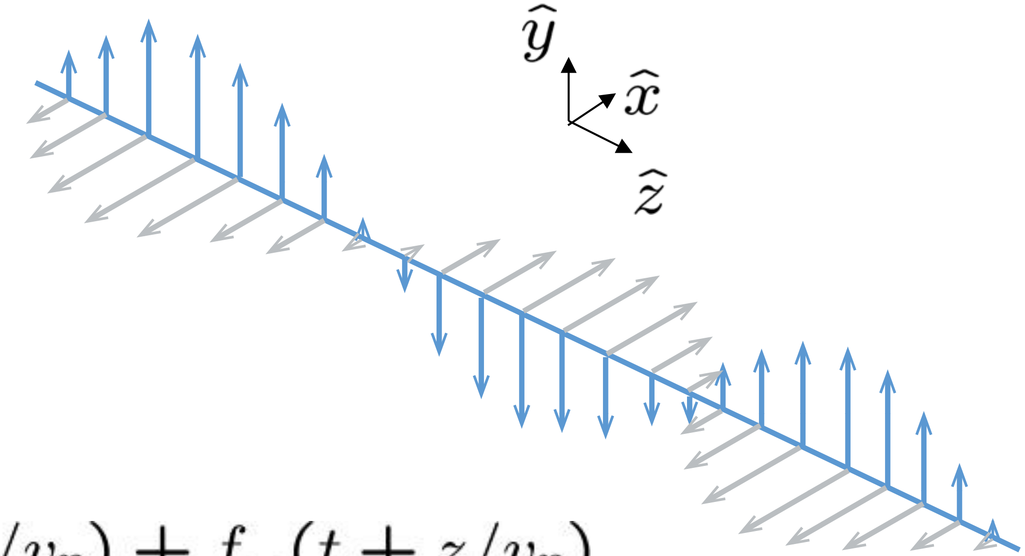
In vacuum...

$$E_y = f_+(t - z/c) + f_-(t + z/c)$$

$$H_x = -\sqrt{\frac{\epsilon_o}{\mu_o}} \Big(f_+(t - z/c) - f_-(t + z/c) \Big)$$

A Uniform Plane Wave

$$\frac{\partial B_x(z_o)}{\partial z} = \epsilon\mu \frac{\partial E_y}{\partial t}$$



Inside a material…

$$E_y = f_+(t - z/v_p) + f_-(t + z/v_p)$$

$$H_x = -\sqrt{\frac{\epsilon}{\mu}}\left(f_+(t - z/v_p) - f_-(t + z/v_p)\right)$$

… where $$v_p = \frac{1}{\sqrt{\mu\epsilon}}$$ is known as the phase velocity of the wave

The Characteristic Impedance

$$H_x = -\sqrt{\frac{\epsilon}{\mu}}\left(f_+(t - z/v_p) - f_-(t + z/v_p)\right)$$

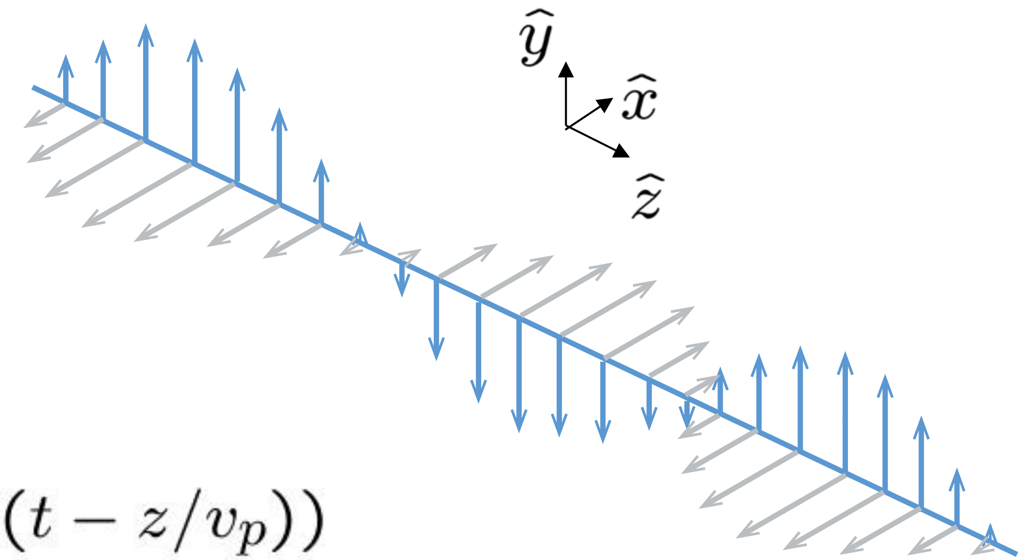
- η is the ***intrinsic impedance*** of the medium given by

$$\eta = \sqrt{\frac{\mu}{\epsilon}}$$

- Like the velocity of propagation, the intrinsic impedance is independent of the source and is determined only by the properties of the medium.

$$\eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}} \approx 377 \text{ Ohms}$$

Sinusoidal Uniform Plane Waves



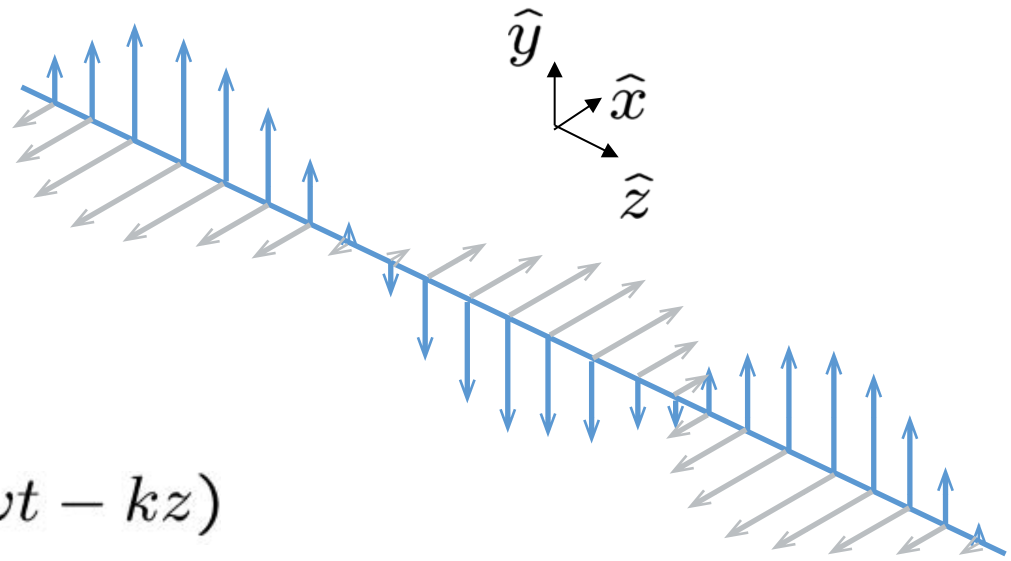
$$f_+(t - z/v_p) = A\cos\left(\omega\left(t - z/v_p\right)\right)$$

$$= A\cos\left(\omega t - kz\right)$$ ... where $k = \dfrac{\omega}{v_p}$ ... is known as the wave-number

$$f_-(t + z/v_p) = A\cos\left(\omega\left(t + z/v_p\right)\right)$$

$$= A\cos\left(\omega t + kz\right)$$

Sinusoidal Uniform Plane Waves



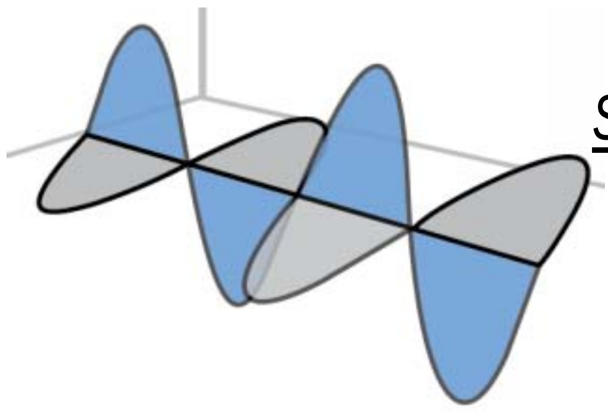
$$f_+(t - z/c) = A\cos(\omega t - kz)$$

$$f_-(t + z/c) = A\cos(\omega t + kz)$$

$$k = \frac{\omega}{c}$$

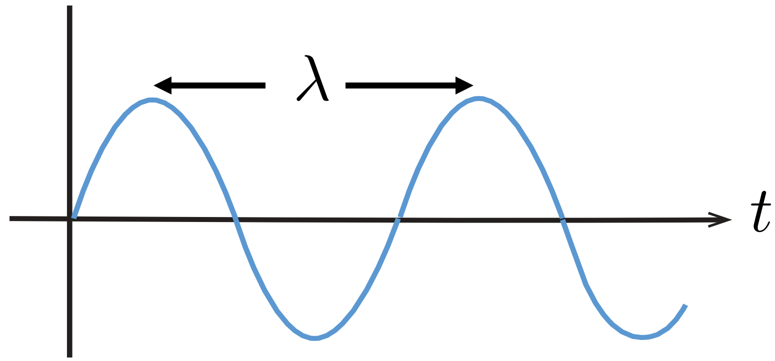
$$E_y = A_1 cos(\omega t - kz) + A_2 cos(\omega t + kz)$$

$$H_x = -\frac{A_1}{\eta} cos(\omega t - kz) + \frac{A_2}{\eta} cos(\omega t + kz)$$



Sinusoidal Uniform Plane Waves

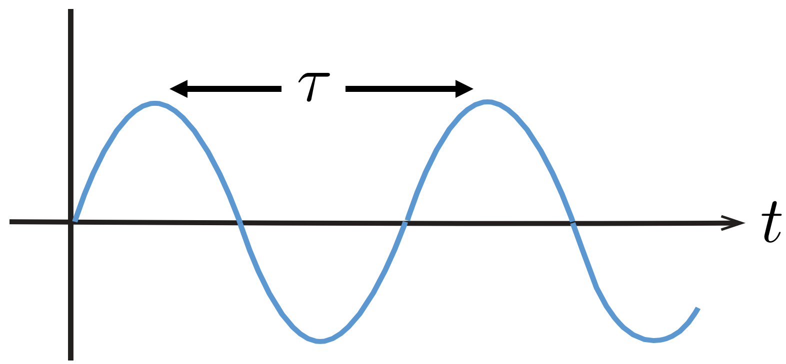
Spatial quantities:



$$k = \frac{\omega}{v_p}$$

$$k = \frac{2\pi}{\lambda}$$

Temporal quantities:



$$\omega = \frac{2\pi}{\tau}$$

$$E_y = A_1 \cos(\omega t - kz) + A_2 \cos(\omega t + kz)$$

$$H_x = -\frac{A_1}{\eta} \cos(\omega t - kz) + \frac{A_2}{\eta} \cos(\omega t + kz)$$

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How Are Uniform EM Plane Waves Launched?

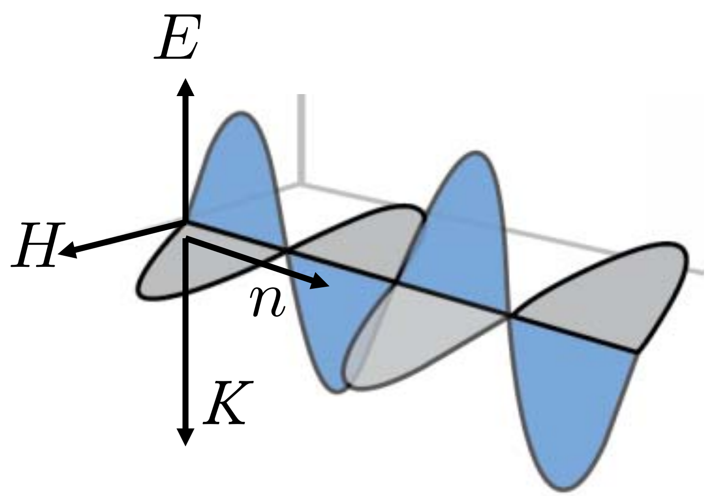
Generally speaking, electromagnetic waves are launched by time-varying charge distributions and currents, that together must satisfy:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

$$\frac{\partial J}{\partial x} + \frac{\partial J}{\partial y} + \frac{\partial J}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$

Man-made systems that launch waves are often called antennas. Uniform plane waves are launched by current sheets:

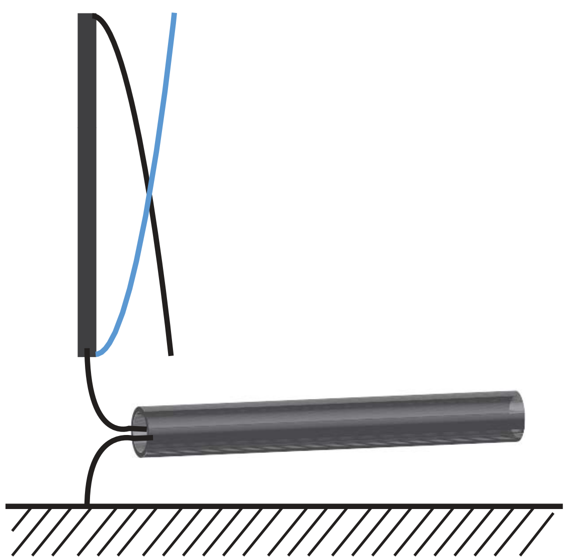
$$\hat{n} \times \vec{H} = \vec{K}$$

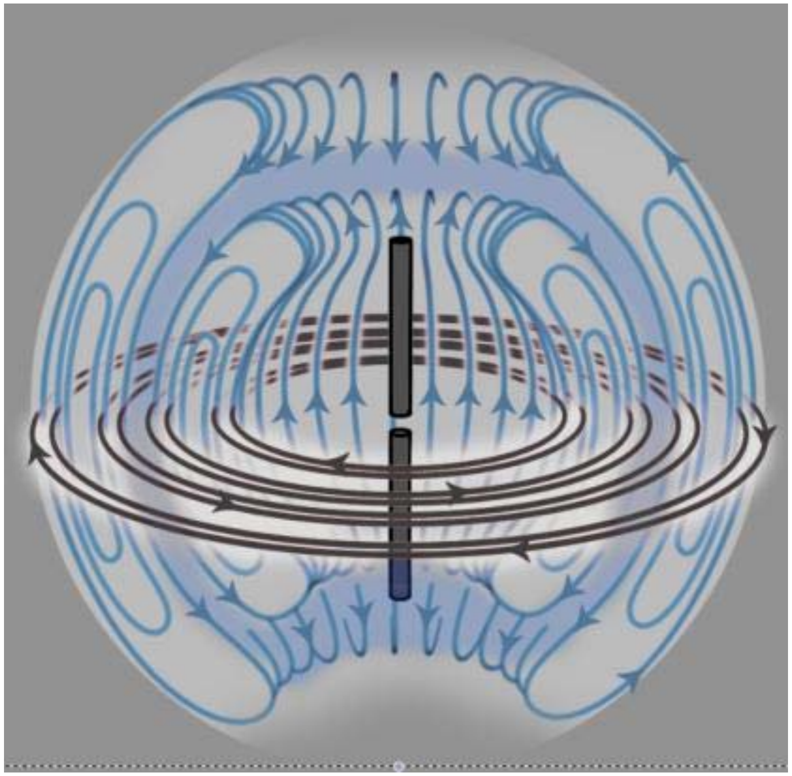


Dipole Antenna

<u>Quarter wavelength vertical antenna</u> has one connection to the vertical element and uses earth connection to provide an image for the other quarter wave. The voltage and current waveforms are out of phase.

The antenna generates (or receives) the omni-directional radiation pattern in the horizontal plane. The antenna does not have to be re-orientated to keep the signals constant as, for example, a car moves its position.





Electric fields (blue) and magnetic fields (gray) radiated by a dipole antenna

KEY TAKEAWAYS

Time-varying E_y generates spatially varying B_z

$$\frac{\partial^2 B_x(z_o)}{\partial z\,\partial t} = \epsilon_o \mu_o \frac{\partial^2 E_y}{\partial t^2}$$

Time-varying B_z generates spatially varying E_y

$$\frac{\partial^2 E_y}{\partial z^2} = \frac{\partial^2 B_x(z_o)}{\partial t\,\partial z}$$

$$\frac{\partial^2 E_y}{\partial z^2} = \epsilon_o \mu_o \frac{\partial^2 E_y}{\partial t^2}$$

The1-D Wave Equation has solutions of the form

$$E_y = f_+(t - z/c) + f_-(t + z/c)$$

... with propagation velocity: (speed of light)
$$c = \frac{1}{\sqrt{\epsilon_o \mu_o}}$$

... and more generally: (phase velocity)
$$v_p = \frac{1}{\sqrt{\mu \epsilon}}$$

$$\eta_o = \sqrt{\frac{\mu_o}{\epsilon_o}} \approx 377 \text{ Ohms}$$

$$E_y = A_1 cos(\omega t - kz) + A_2 cos(\omega t + kz)$$

$$H_x = -\frac{A_1}{\eta} cos(\omega t - kz) + \frac{A_2}{\eta} cos(\omega t + kz)$$

... where $\eta = \sqrt{\frac{\mu}{\epsilon}}$ ... is known as the intrinsic impedance

$k = \frac{\omega}{v_p} = \frac{2\pi}{\lambda}$ ... is known as the wave-number

6.007 Electromagnetic Energy: From Motors to Lasers
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