Complex Numbers and Phasors

Reading - Shen and Kong - Ch. 1

Outline

Linear Systems Theory
Complex Numbers
Polyphase Generators and Motors
Phasor Notation

True / False

1. In Lab 1 you built a motor about 5 cm in diameter. If this motor spins at 30 Hz, it is operating in the quasi-static regime. ____

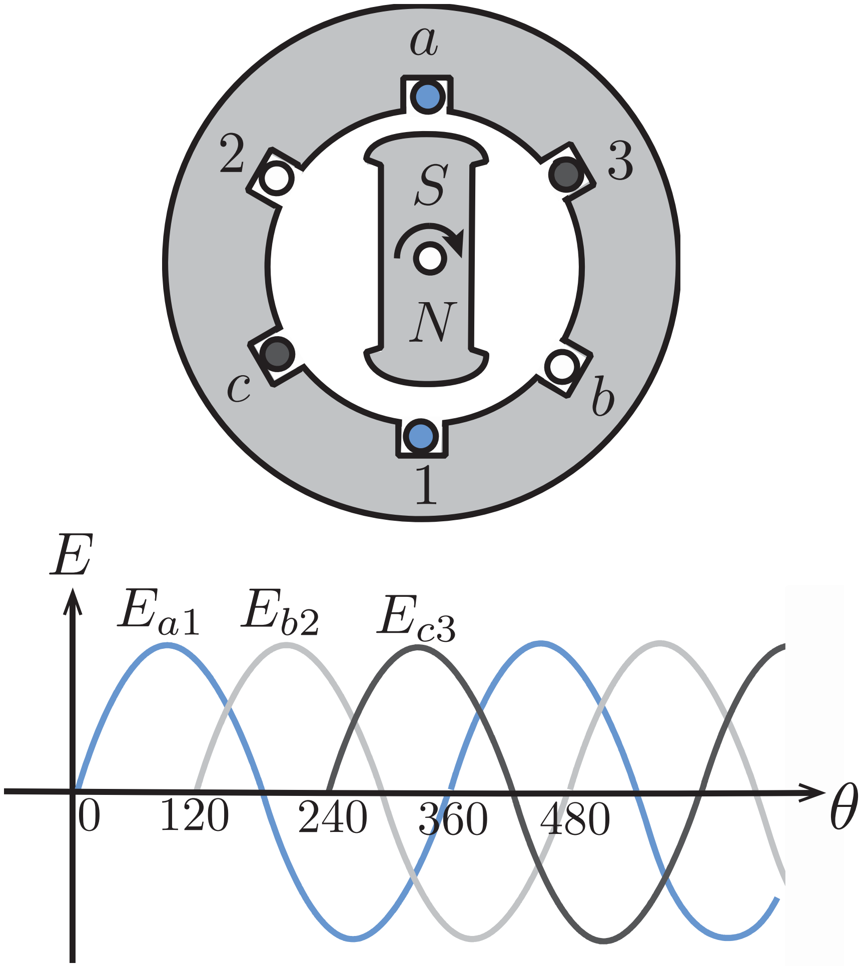
2. The wave number k (also called the wave vector) describes the "spatial frequency" of an EM wave. ____

3. This describes a 1D propagating wave:

$$A_0 cos(\omega t) cos(kz)$$

____

Electric Power System



The electric power grid operates at either 50 Hz or 60 Hz, depending on the region.

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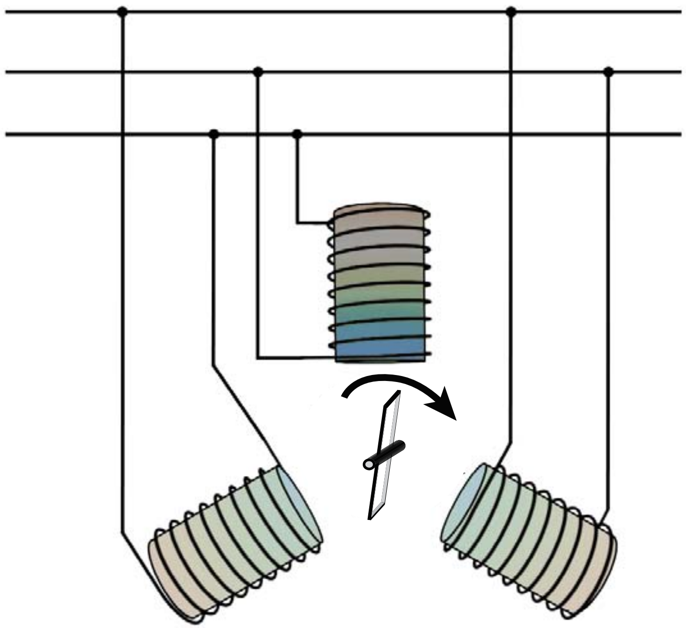
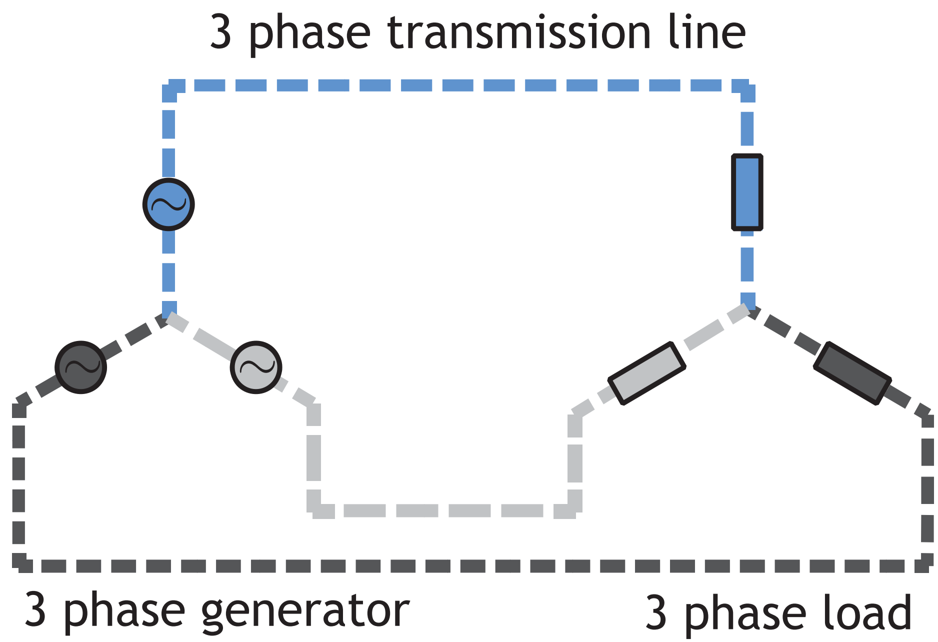
Electric Power System





The Challenge of Sinusoids

Models of dynamic systems couple time signals to their time derivatives. For example, consider the system

$$T\frac{dy(t)}{dt} + y(t) = x(t)$$

Where T is a constant. Suppose that $x(t)$ is sinusoidal, then $y(t)$ and its time derivative will take the form

$$y(t) \sim sin(\omega t + \phi)$$

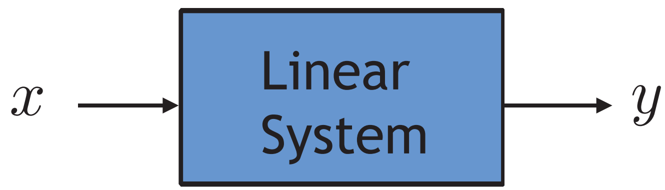
$$\frac{dy(t)}{dt} \sim cos(\omega t + \phi)$$

Coupling the signal $y(t)$ to its time derivative will involve trigonometric identities which are cumbersome! Are there better analytic tools? (Yes, for linear systems.)

Linear Systems

Homogeneity

If



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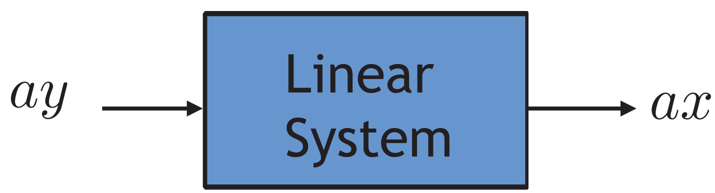
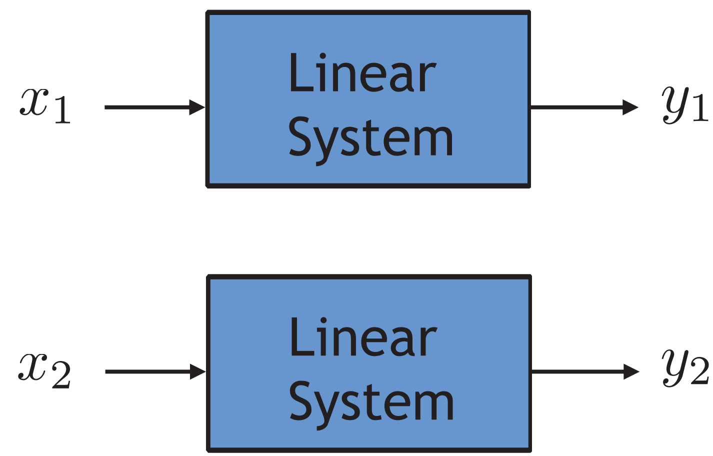


Superposition

If



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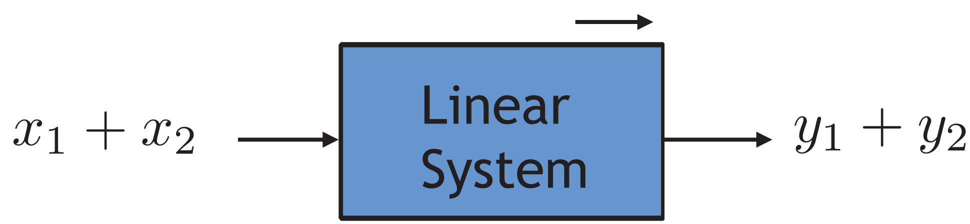
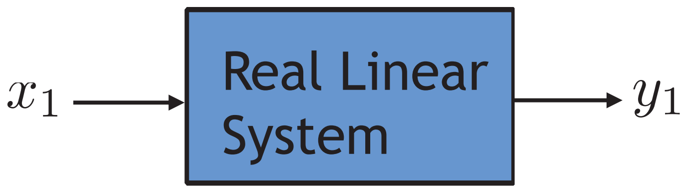
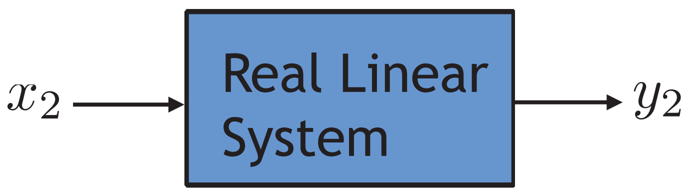


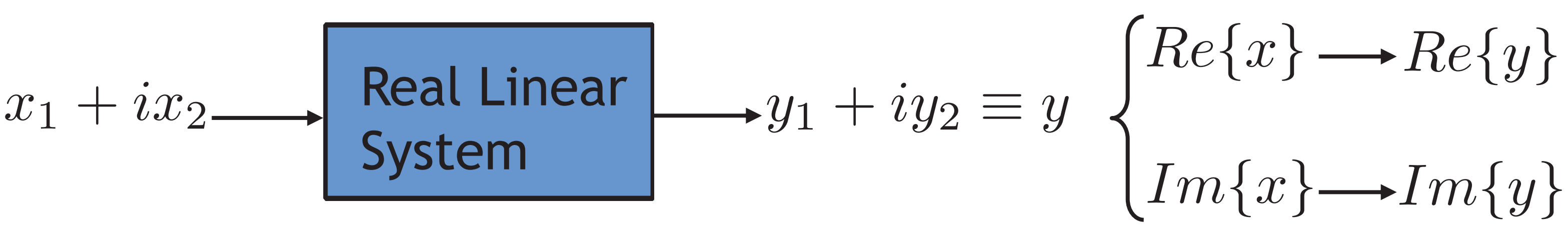
Linear Systems

If



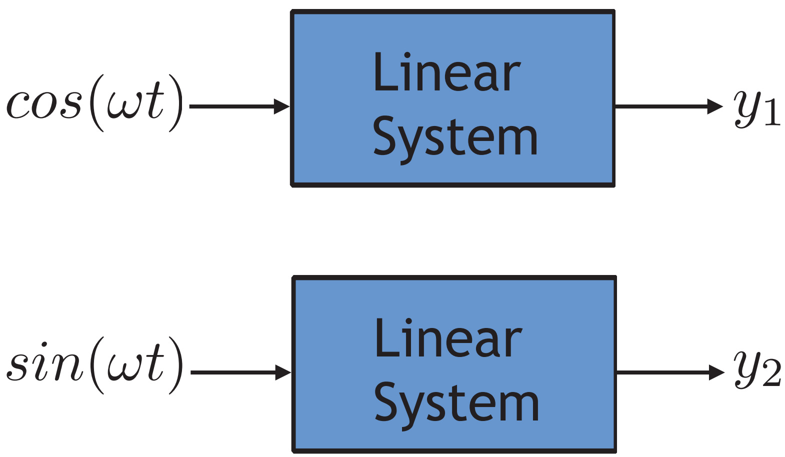


then



Linear Systems

If



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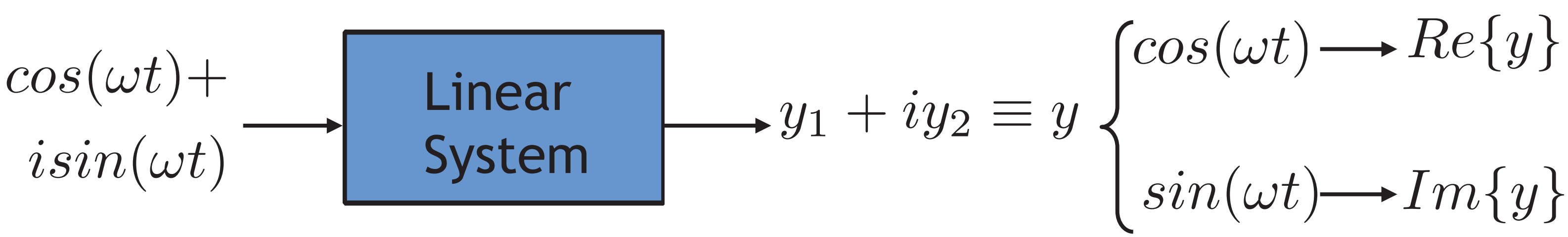

Now Responses to Sinusoids are Easy

$$cos(\omega t) + isin(\omega t) = e^{i\omega t}$$

… Euler's relation

$$\frac{d}{dt} e^{i\omega t} = iwe^{i\omega t}$$

Combining signals with their time derivatives, both expressed as complex exponentials, is now much easier.

Analysis no longer requires trigonometric identities. It requires only the manipulation of complex numbers, and complex exponentials!

Imaginary numbers

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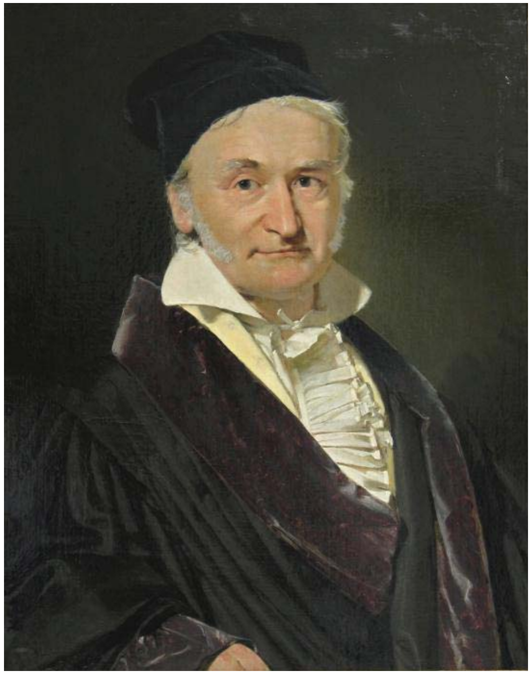
Gerolamo Cardano
(1501-1576)

- Trained initially in medicine
- First to describe typhoid fever
- Made contributions to algebra

- 1545 book *Ars Magna* gave solutions for cubic and quartic equations (cubic solved by Tartaglia, quartic solved by his student Ferrari)
- First Acknowledgement of complex numbers

Descartes coined the term “imaginary” numbers in 1637

The work of Euler and Gauss made complex numbers more acceptable to mathematicians



Notation

Image is in the public domain.

$$z = x + iy$$

Complex numbers in mathematics

Euler, 1777

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$$F = e_1 + je_2$$

Analysis of alternating current in electrical engineering

Steinmetz, 1893

Complex Numbers (Engineering convention)

We define a **complex** number with the form

$$z = x + iy$$

Where x , y are real numbers.

The **real part** of z, written $Re\{z\}$ is x.

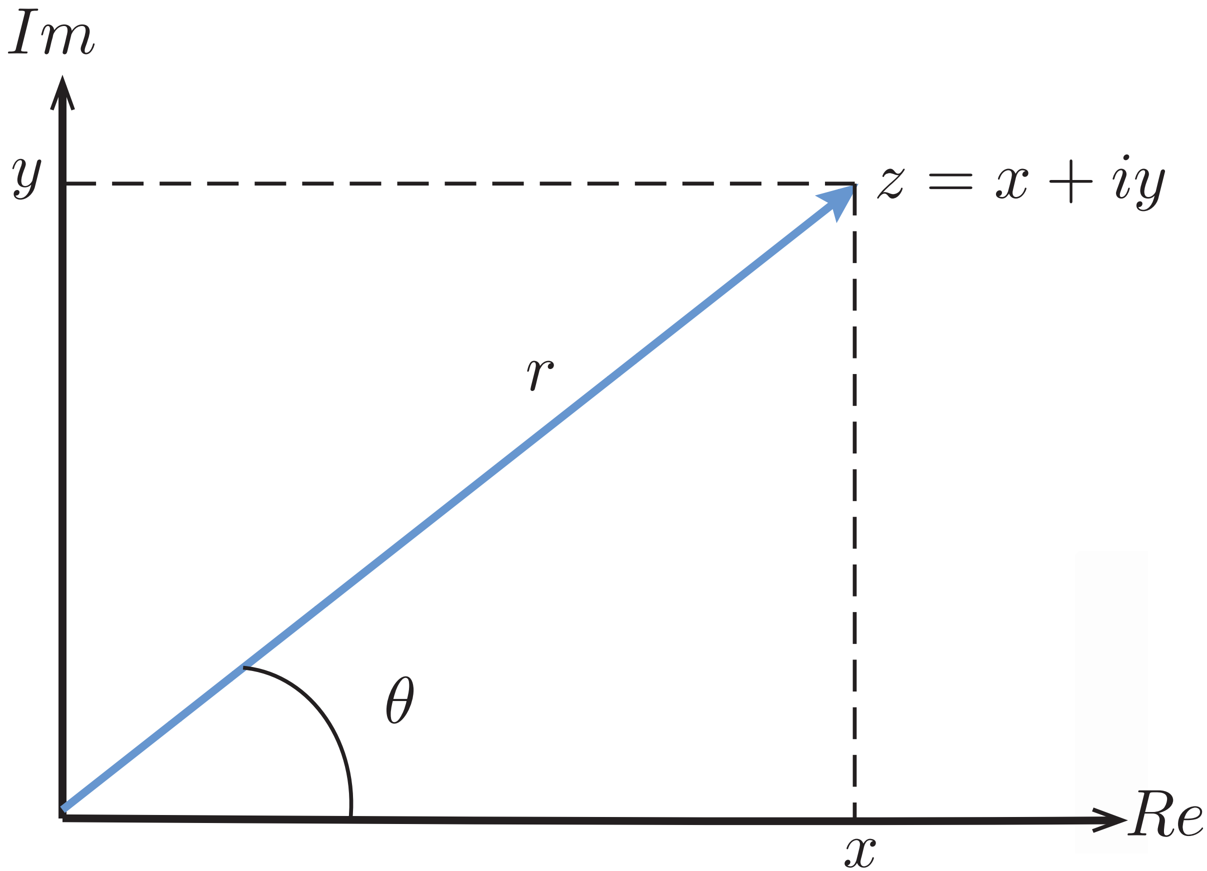
The **imaginary part** of z, written $Im\{z\}$, is y.

- Notice that, confusingly, the imaginary part is a real number.

So we may write z as

$$\boxed{z = Re\{z\} + iIm\{z\}}$$

Complex Plane



$x = r cos(\theta)$ and $y = r sin(\theta)$

$$x + iy = r(cos(\theta) + i sin(\theta)) = re^{i\theta}$$

Polar Coordinates

In addition to the Cartesian form, a complex number z may also be represented in **polar form**:

$$z = re^{i\theta}$$

Here, r is a real number representing the magnitude of z, and θ represents the angle of z in the complex plane.

Multiplication and division of complex numbers is easier in polar form:

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Addition and subtraction of complex numbers is easier in Cartesian form.

Converting Between Forms

To convert from the Cartesian form $z = x + iy$ to polar form, note:

$$zz^* = (re^{i\theta})(re^{-i\theta}) = r^2 \longrightarrow \boxed{r = \mid z \mid = \sqrt{x^2 + y^2}}$$

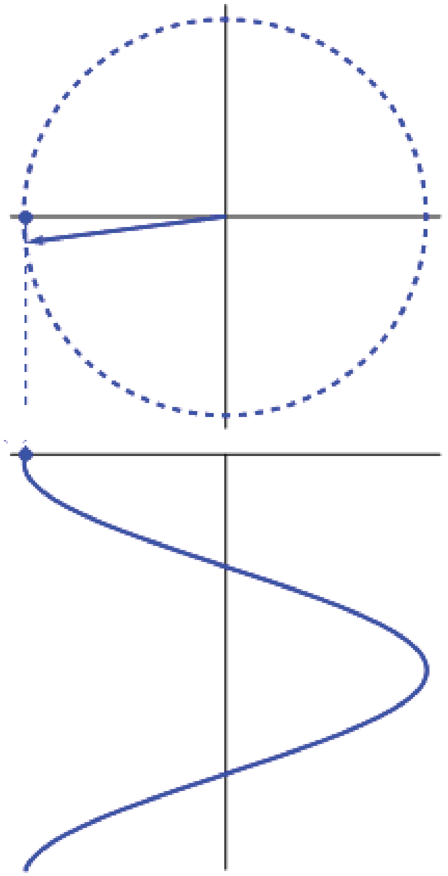
$$z = re^{i\theta} = r(cos(\theta) + isin(\theta)) = x + iy$$

$$\boxed{x = rcos(\theta)}$$

$$\boxed{y = rsin(\theta)}$$

$$\frac{y}{x} = \frac{rsin(\theta)}{rcos(\theta)} = tan(\theta) \longrightarrow \boxed{\theta = tan^{-1}(\frac{y}{x})}$$

Phasors

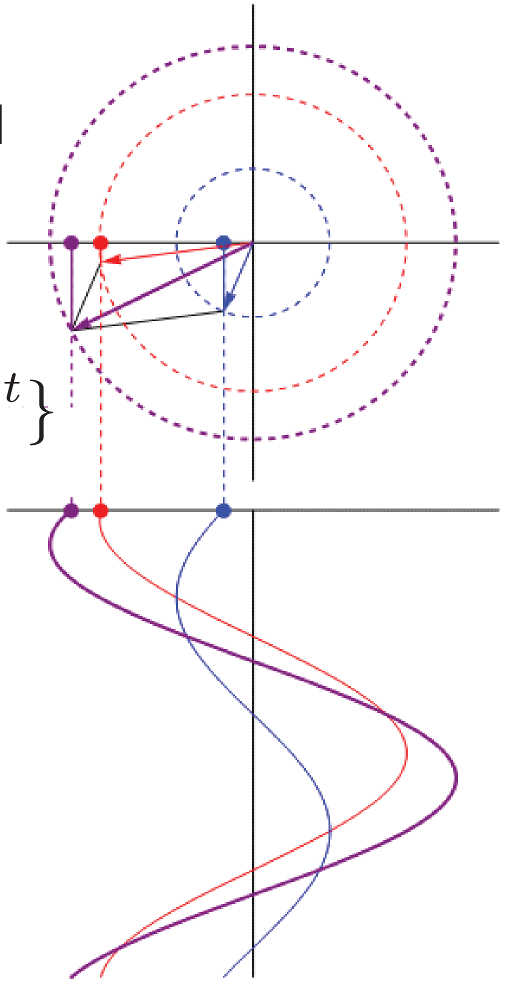


A phasor, or phase vector, is a representation of a sinusoidal wave whose amplitude (A), phase (θ), and frequency (ω) are time-invariant.

$$A_0 cos(\omega t + \theta) = Re\{A_0 e^{i\theta} e^{i\omega t}\}$$

The phasor spins around the complex plane as a function of time.

Phasors of the same frequency (ω) can be added.

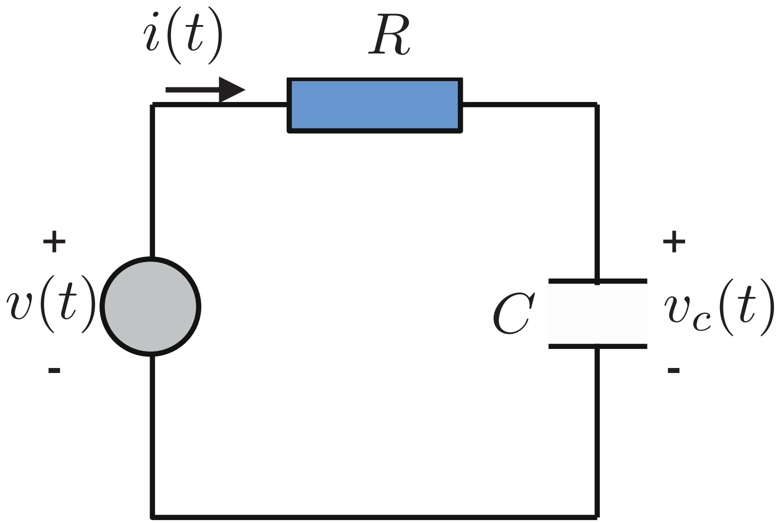


This is an animation
But it's a known fact

Modern Version of Steinmetz' Analysis

1. Begin with a time-dependent analysis problem posed in terms of real variables.
2. Replace the real variables with variables written in terms of complex exponentials; $e^{i\omega t}$ is an eigenfunction of linear time-invariant systems.
3. Solve the analysis problem in terms of complex exponentials.
4. Recover the real solution from the results of the complex analysis.

Example: RC Circuit



Assume that the drive is sinusoidal:

$$v(t) = V_0 cos(\omega t)$$

And solve for the current $i(t)$

$$\begin{cases} v(t) = Ri(t) + v_c(t) \\ i(t) = C\frac{d}{dt}v_c(t) \end{cases}$$

$$\frac{d}{dt}i(t) + \frac{i(t)}{RC} = \frac{1}{R}\frac{d}{dt}v(t)$$

Use Steinmetz AC method

Sinusoidal voltage source expressed in terms of complex exponential

$$v(t) \;=\; V_0 \cos(\omega t) \;=\; \mathrm{Re}\left\{ V_0 e^{j\omega t} \right\}$$

Complex version of problem

$$i(t) \;=\; \mathrm{Re}\left\{ I_0 e^{j\omega t} \right\} \qquad\qquad j\omega I_0 + \frac{I_0}{RC} \;=\; \frac{j\omega V_0}{R}$$

Recover real solution from complex problem

$$i(t) \;=\; \mathrm{Re}\left\{ I_0 e^{j\omega t} \right\} \;=\; \mathrm{Re}\left\{ \frac{1}{R + \dfrac{1}{j\omega C}} V_0 e^{j\omega t} \right\}$$

Natural Response / Homogeneous Solution

Linear constant-coefficient ordinary differential equations of the form

$$\frac{d^n y(t)}{dt^n} + A_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + ...A_1\frac{dy(t)}{dt} + A_0 y(t) = 0$$

have solutions of the form $y(t) \sim e^{st}$ where

$$s^n + A_{n-1}s^{n-1} + ...A_1 s + A_0 = 0$$

Can we always find the roots of such a (characteristic) polynomial?

Polynomial Roots

Can we always find roots of a polynomial? The equation

$$x^2 + 1 = 0$$

has no solution for x in the set of real numbers. If we define, and then use, a number that satisfies the equation

$$x^2 = -1$$

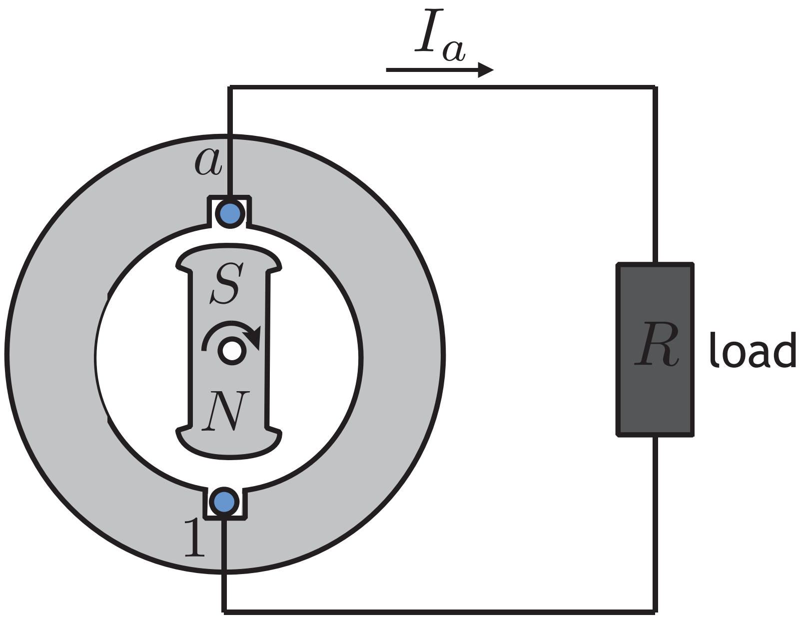
that is,

$$x = \sqrt{-1} \text{ or } x = -\sqrt{-1}$$

then we can always find the n roots of a polynomial of degree n.

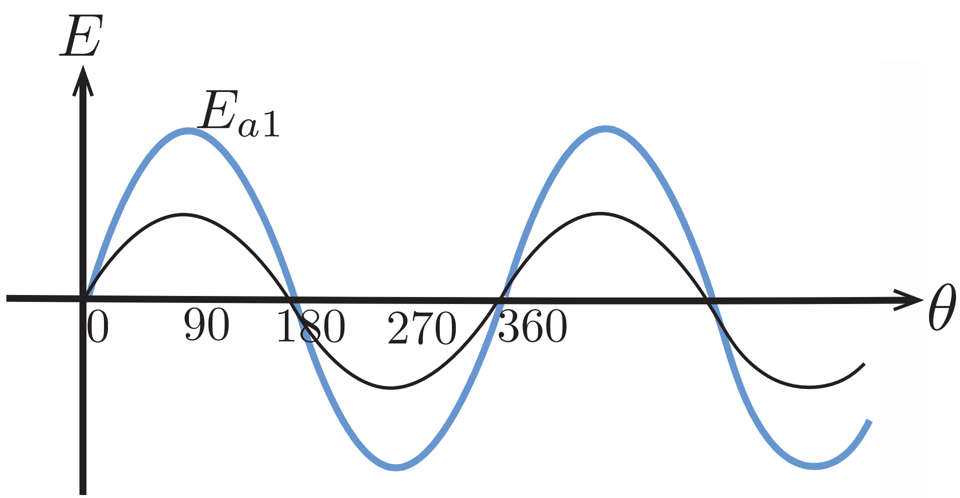
Complex roots of a characteristic polynomial are associated with an oscillatory (sinusoidal) natural response.

Single-phase Generator

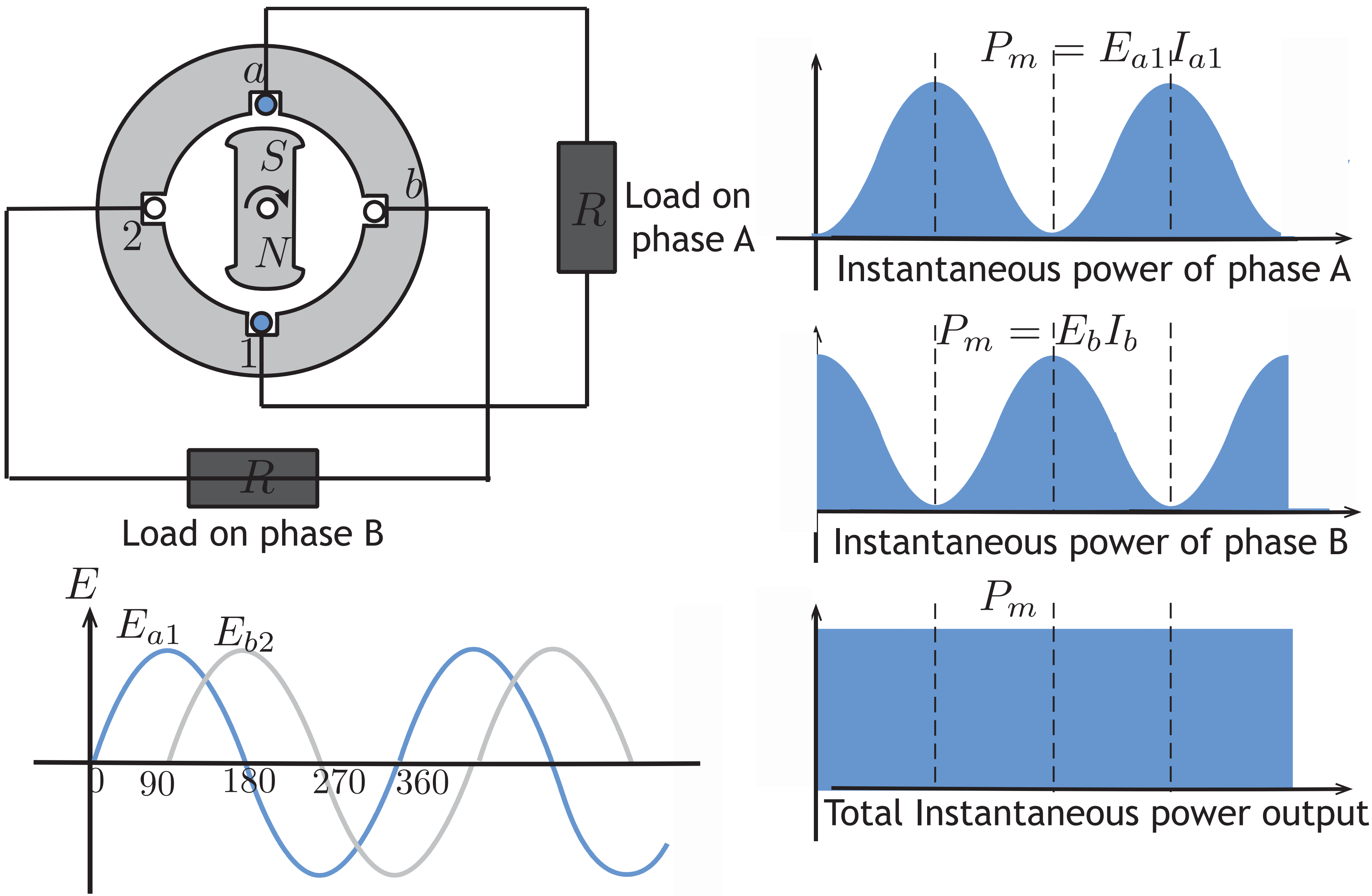


$$P_m = E_{a1} I_{a1}$$

Instantaneous power of phase A

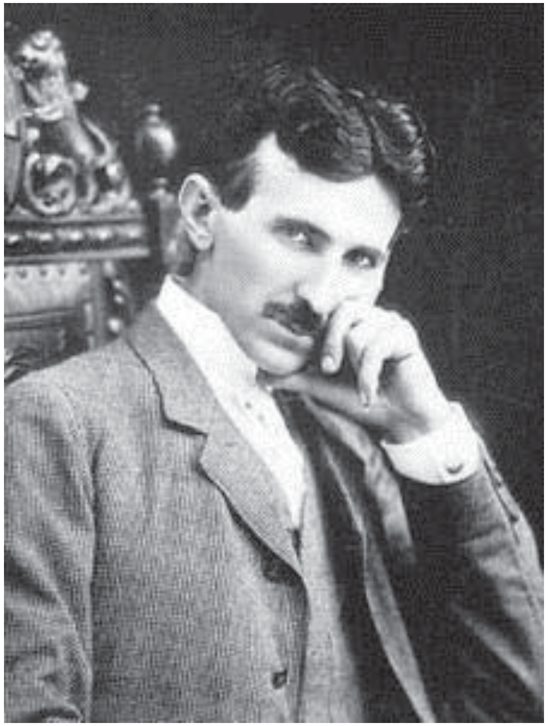


Two-phase Generator

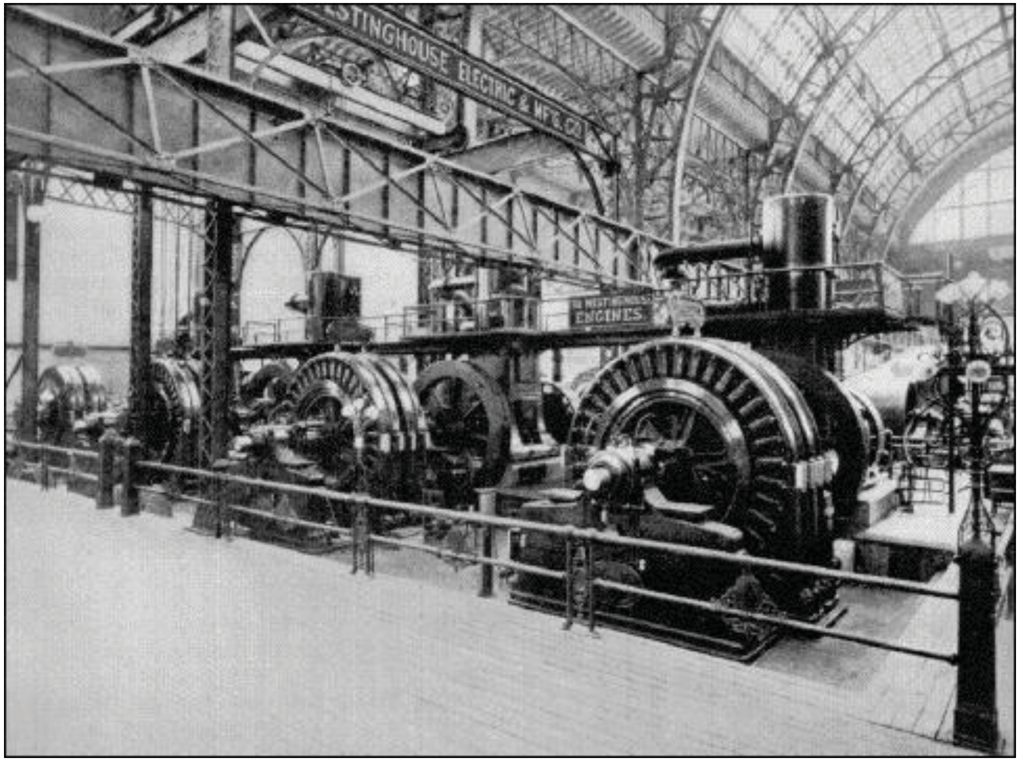


Patented two-phase electric motor

Some people mark the introduction of
Tesla's two-phase motor as the beginning
of the second industrial revolution (concept 1882, patent, 1888)

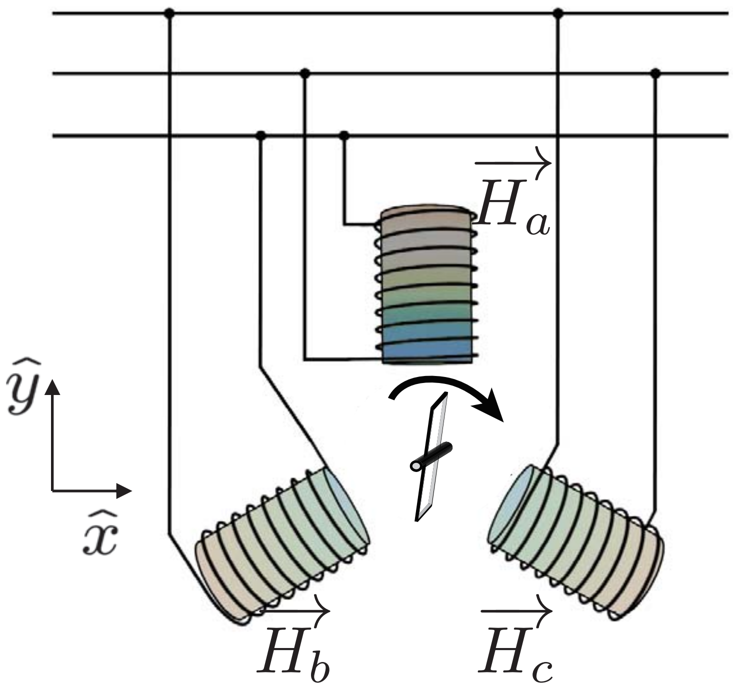


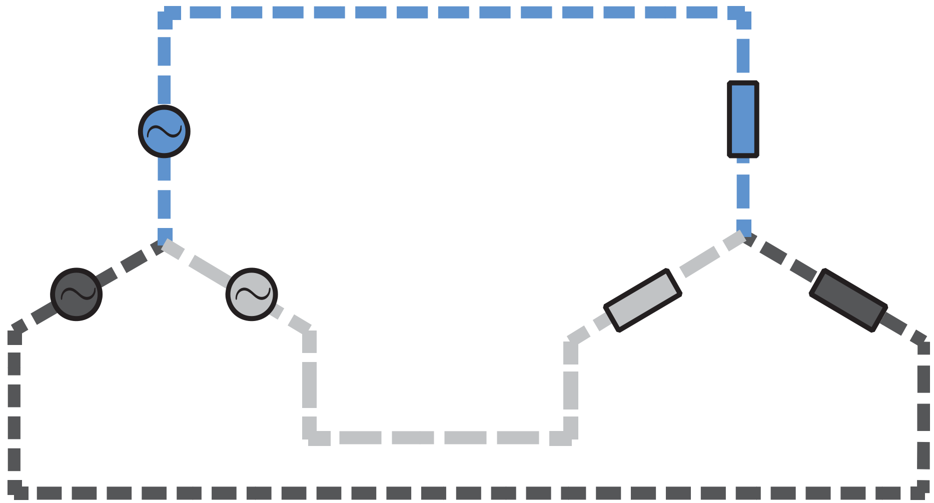
Nikola Tesla circa 1886



AC generators used to light the Chicago
Exposition in 1893

Electric Power System … Revisited





What about Space?

Maxwell's equations are partial differential equations, and hence involve "signals" that are functions of space and their spatial derivates. Correspondingly, we will find complex exponential functions of the form

$$e^{ikx}$$

to be very useful in analyzing dynamic systems described with Maxwell's equations.

What's the Difference between i and j ?

Engineering

$$E = E_0 e^{j(\omega t - kx)}$$

Physics

$$E = E_0 e^{-i(\omega t - kx)}$$

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Can go back and forth between physics and engineering literature
If we adopt the convention

$$j = -i$$

We will ultimately use both notations in 6.007

6.007 Electromagnetic Energy: From Motors to Lasers
Spring 2011