Electromagnetic Waves in Materials

Outline

Review of the Lorentz Oscillator Model

Complex index of refraction – what does it mean?

TART

Microscopic model for plasmas and metals

True / False

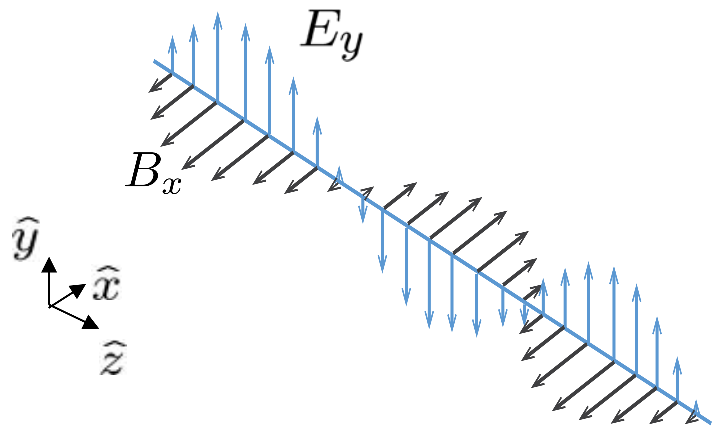
1. In the Lorentz oscillator model of an atom, the electron is bound to the nucleus by a spring whose spring constant is the same for any atom. ____

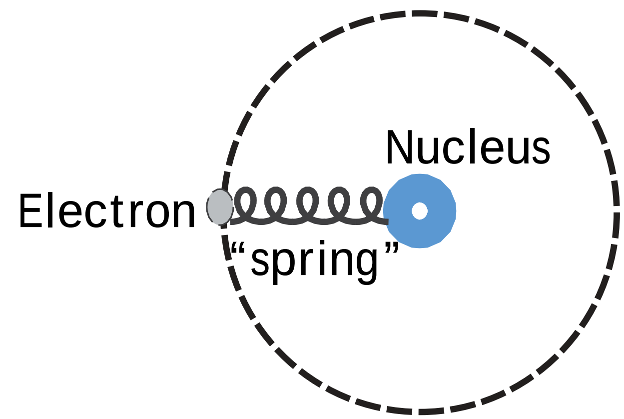
2. The following is the differential equation described by the Lorentz oscillator model: ____

$$m\frac{d^2y}{dt^2} = -m\omega_o^2 y - m\gamma\frac{dy}{dt}$$

3. For dielectrics, we can approximate the index of refraction as the square root of the dielectric constant. ____

Microscopic Description of Dielectric Constant





$$m\frac{d^2y}{dt^2} = -m\omega_o^2 y + qE_y - m\gamma\frac{dy}{dt}$$

Electron mass

Restoring force (binding electron & nucleus)

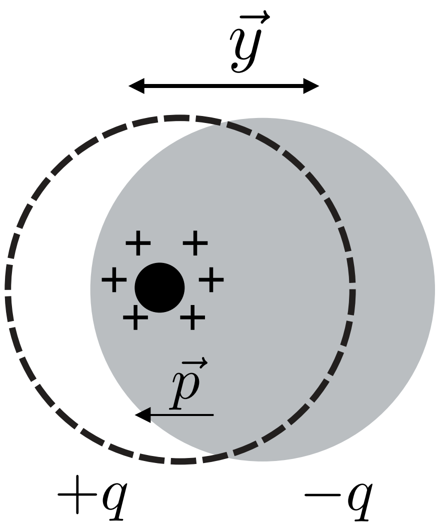
$\vec{E}$ field force

Damping

Solution using complex variables

Lets plug-in the expressions for E_y and y into the differential equation from slide 3:

Natural resonance



$$\frac{d^2}{dt^2}y(t) + \gamma\frac{d}{dt}y(t) + \omega_o^2 y(t) = \frac{q}{m}E_y(t)$$

$$E_y(t) = Re\{E_y e^{j\omega t}\} \qquad y(t) = Re\{y e^{j\omega t}\}$$

$$\omega^2 y + j\omega\gamma y + \omega_o^2 y = \frac{q}{m}E_y$$

$$y = \frac{q}{m}\frac{1}{(\omega_o^2 - \omega^2) + j\omega\gamma}E_y$$

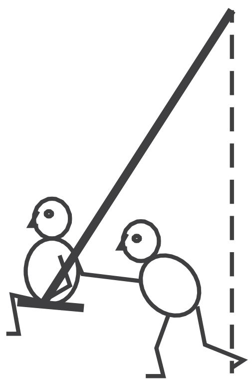
Oscillator Resonance

$$y = \frac{q}{m}\frac{1}{(\omega_0^2 - \omega^2) + j\omega\gamma}E_y$$

$$E_y(t) = Re\{E_y e^{j\omega t}\}$$

$$y(t) = Re\{y e^{j\omega t}\}$$

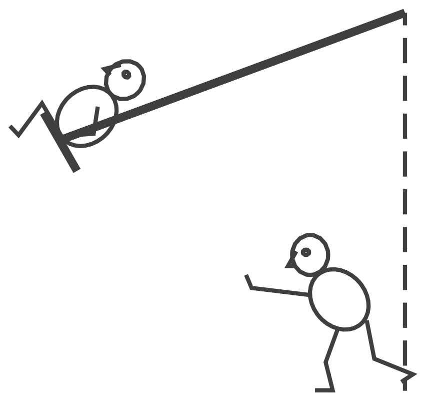
Driven harmonic oscillator: **Amplitude** and **Phase** depend on frequency



Low frequency

medium amplitude

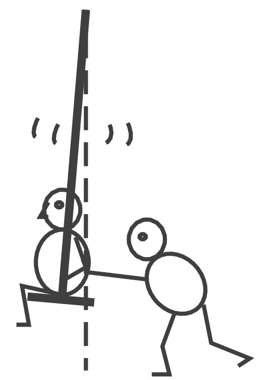
Displacement, y in phase with E_y



At resonance

large amplitude

Displacement, y 90º out of phase with E_y



High frequency

vanishing amplitude

Displacement y and E_y in antiphase

Polarization

Since charge displacement, y, is directly related to polarization, P, of our material we can then rewrite the differential equation:

$$\vec{D} = \epsilon_o \vec{E} + \vec{P}$$

$$P_y = Nqy$$

For linear polarization in $\hat{y}$ direction

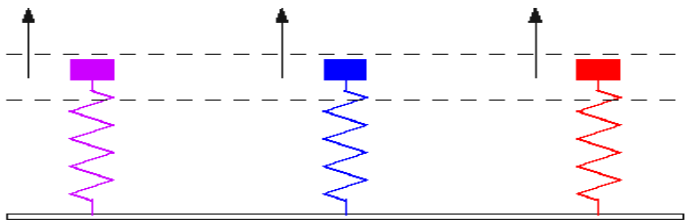
$$(\frac{d}{dt^2} + \gamma \frac{d}{dt} + \omega_o^2) P_y(t) = \frac{Nq^2}{m} E_y(t) = \epsilon_o \omega_p^2 E_y(t)$$

$$\omega_p^2 = \frac{Nq^2}{\epsilon_o m}$$

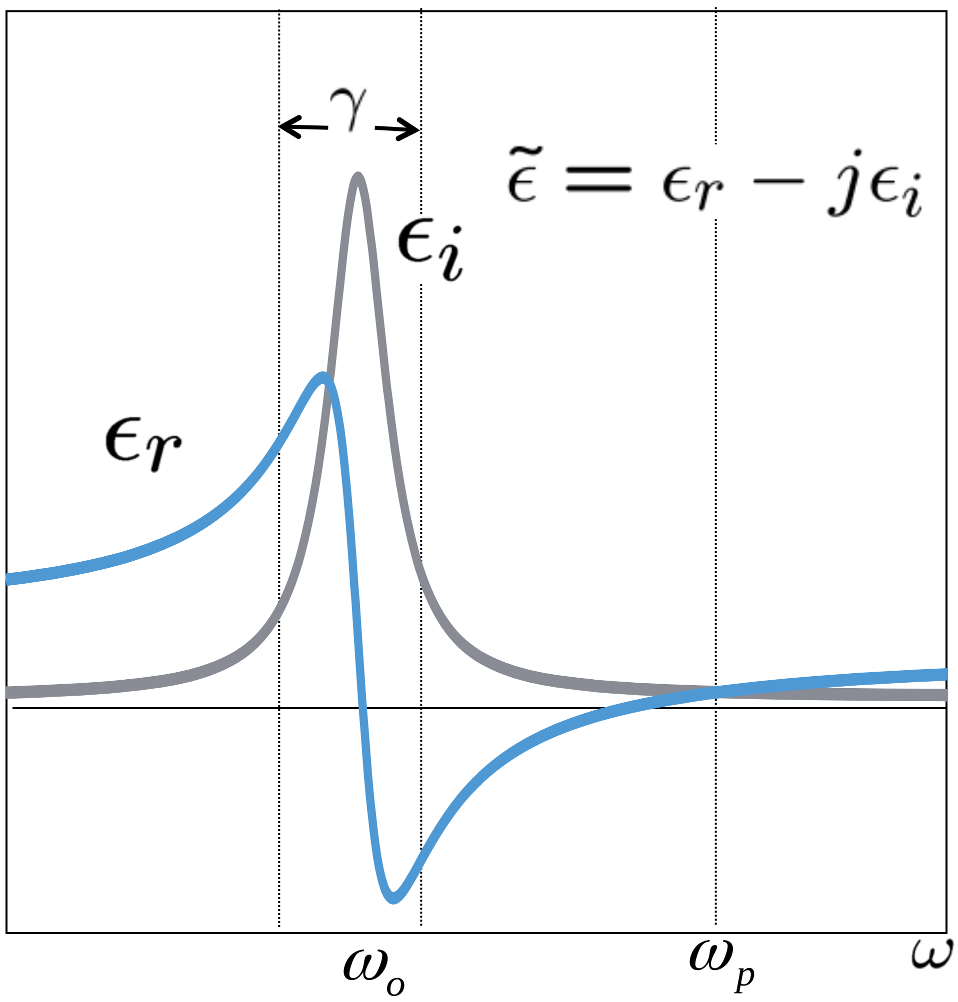
$$P_y(t) = Re\{P_y e^{j\omega t}\}$$

$$P_y = \frac{\omega_p^2}{(\omega_o^2 - \omega^2) + j\gamma\omega} \epsilon_o E_y$$

Microscopic Lorentz Oscillator Model



$$\epsilon = \epsilon_o \left(1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2 + j\omega\gamma}\right)$$

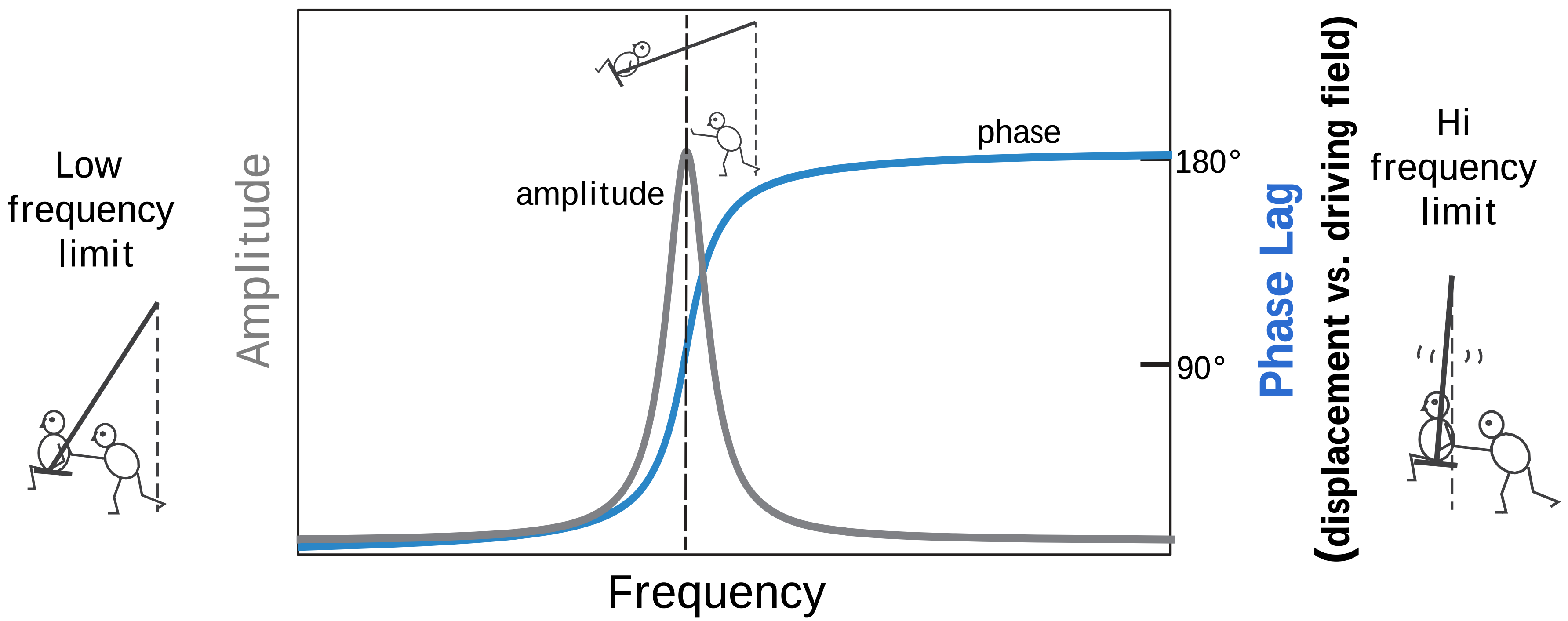


$$\omega_p^2 = \frac{Nq^2}{\epsilon_o m}$$

$$\omega_o^2 = \frac{k_{spring}}{m}$$

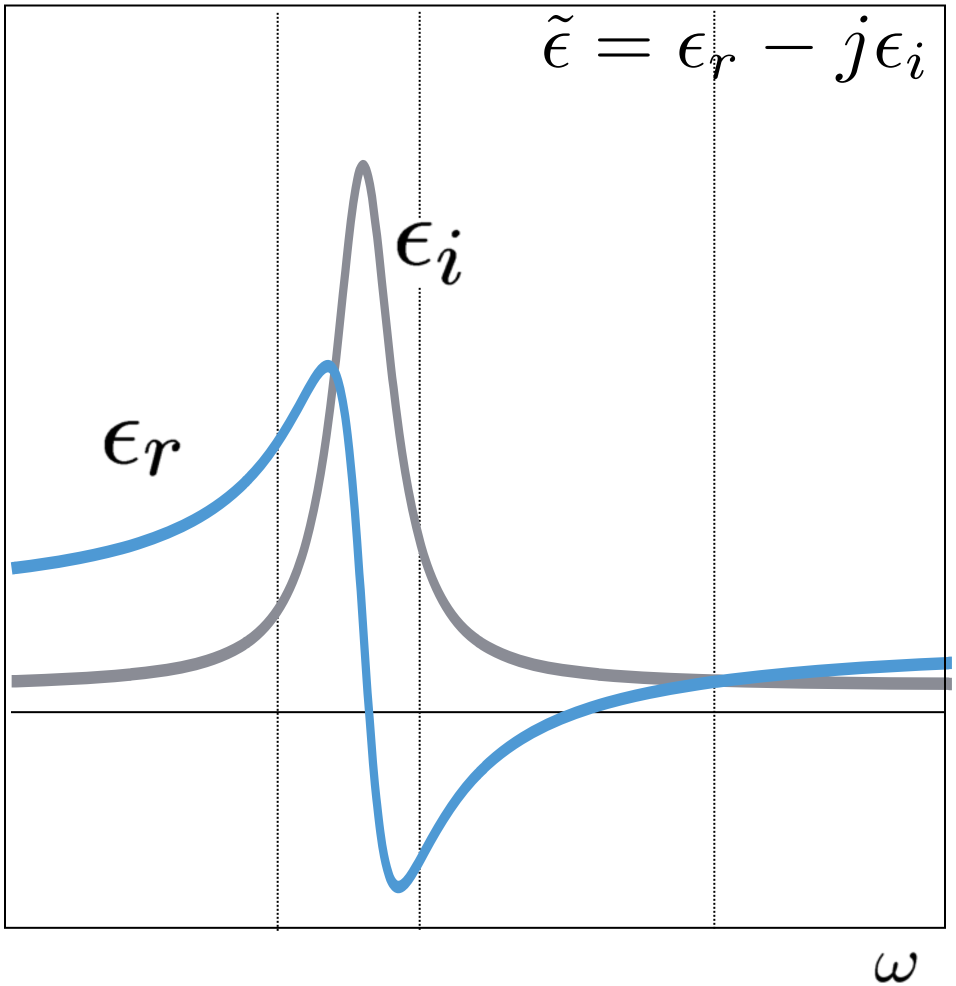
$$\epsilon = \epsilon_o \left(1 + \frac{\omega_p^2}{\omega_o^2 - \omega^2 + j\omega\gamma}\right) \qquad \omega_p^2 = \frac{Nq^2}{\epsilon_o m} \qquad \omega_o^2 = \frac{k_{spring}}{m}$$

Behavior of a driven (and damped) harmonic oscillator can be summarized as follows



This type of response of **bound charges** is typical for many materials

Complex Refractive Index



$$v_p = \frac{1}{\sqrt{\mu\epsilon}}$$

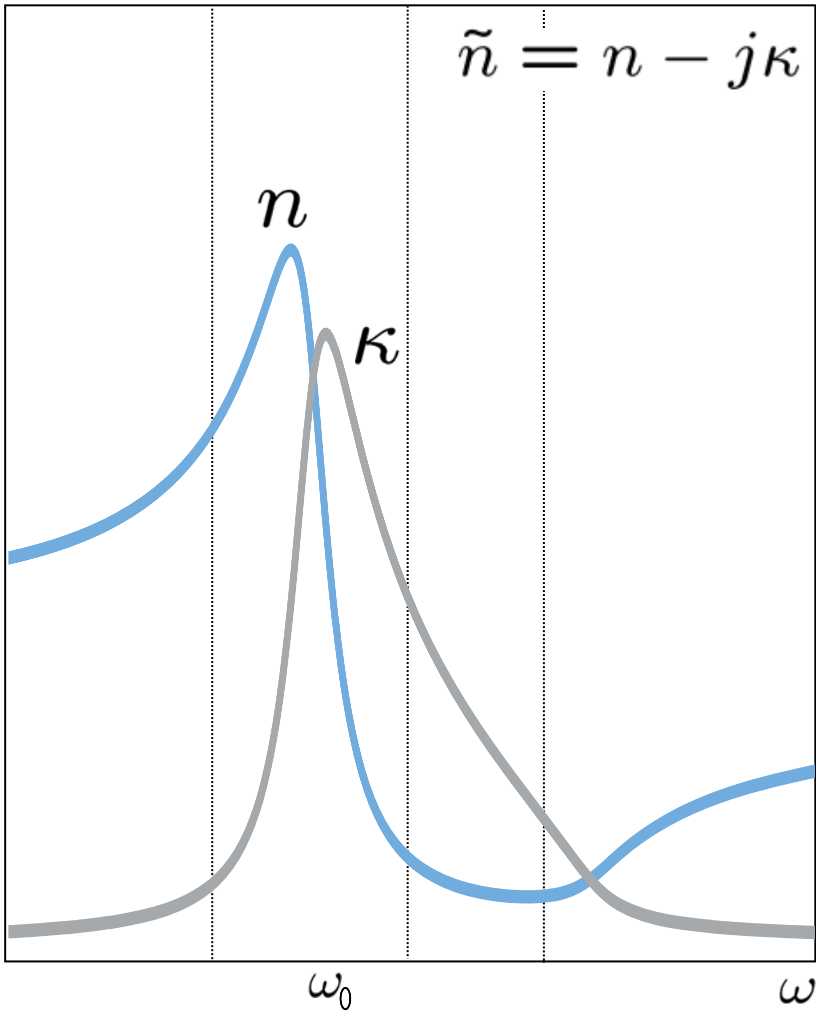
$$n \equiv \frac{c}{v_p} = \frac{\sqrt{\mu\epsilon}}{\sqrt{\mu_o\epsilon_o}} \approx \frac{\sqrt{\epsilon}}{\sqrt{\epsilon_o}}$$

$$\frac{\tilde{\epsilon}}{\epsilon_o} = \tilde{n}^2$$
$$= (n - j\kappa)^2$$
$$= n^2 - \kappa^2 - 2jn\kappa$$

$$\frac{\epsilon_{real}}{\epsilon_o} = n^2 - \kappa^2$$ (real)

$$\frac{\epsilon_{imag}}{\epsilon_o} = 2n\kappa$$ (imaginary)

Absorption Coefficient



$$n = \frac{1}{\sqrt{2}}\sqrt{\epsilon_r + \sqrt{\epsilon_r^2 + \epsilon_i^2}}$$

$$\kappa = \frac{1}{\sqrt{2}}\sqrt{-\epsilon_r + \sqrt{\epsilon_r^2 + \epsilon_i^2}}$$

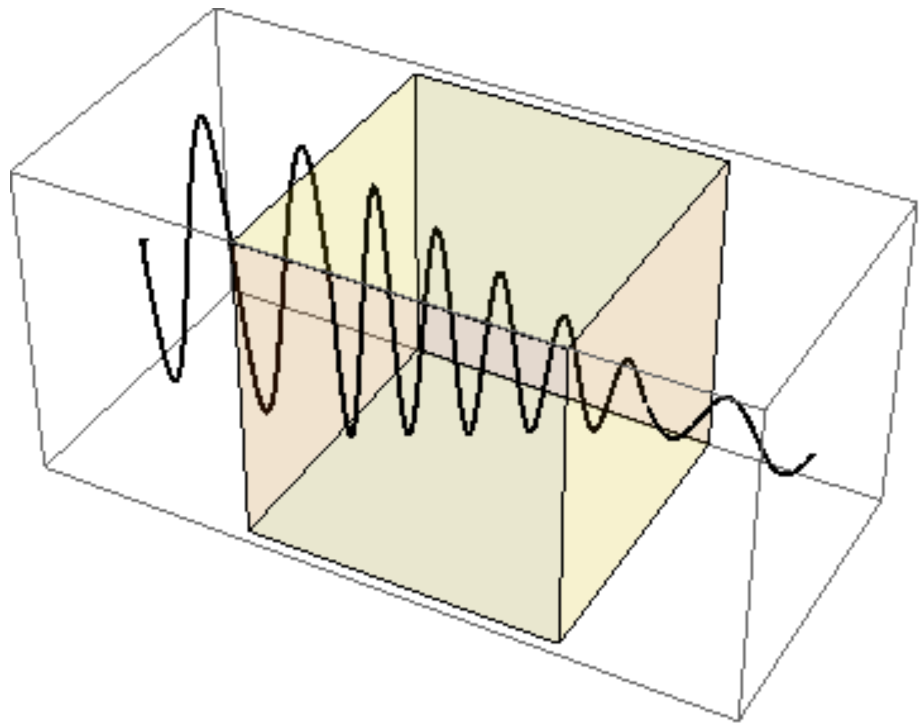
$$E(t,z) = Re\{\tilde{E}_0 e^{-\alpha z/2} e^{j(\omega t - k_0 n z)}\}$$

Absorption

Refractive index

$$\alpha = 2k_o\kappa = 2\frac{2\pi}{\lambda_o}\kappa \quad [\text{cm}^{-1}]$$

Absorption and Reflection



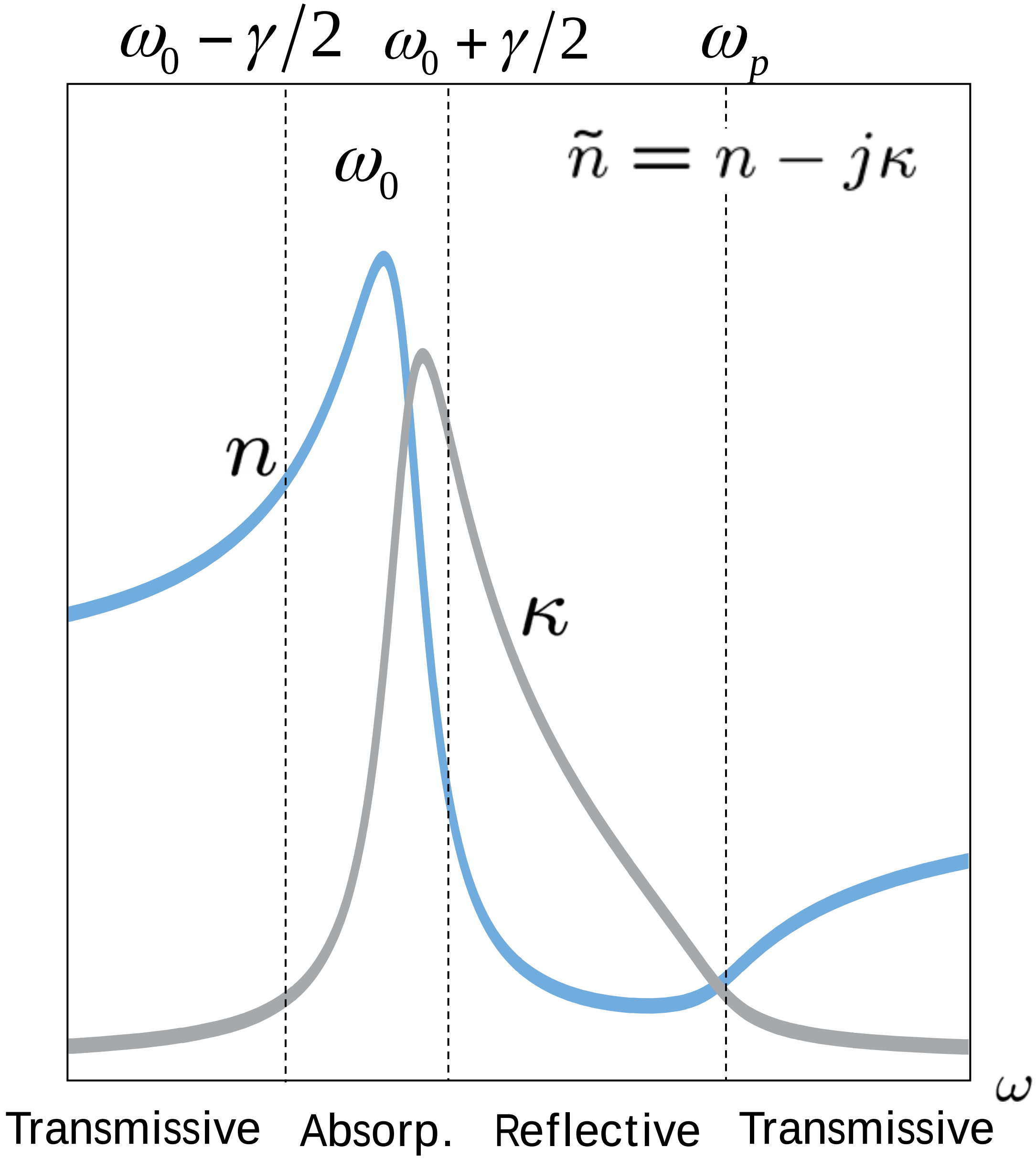
$$E(t,z) = Re\{\tilde{E}_0 e^{-\alpha z/2} e^{j(\omega t - k_0 n z)}\}$$

Absorption coefficient

Refractive index

$$I(z) = I_o e^{-\alpha z}$$ Beer-Lambert Law or Beer's Law

TART



Different resonant frequencies

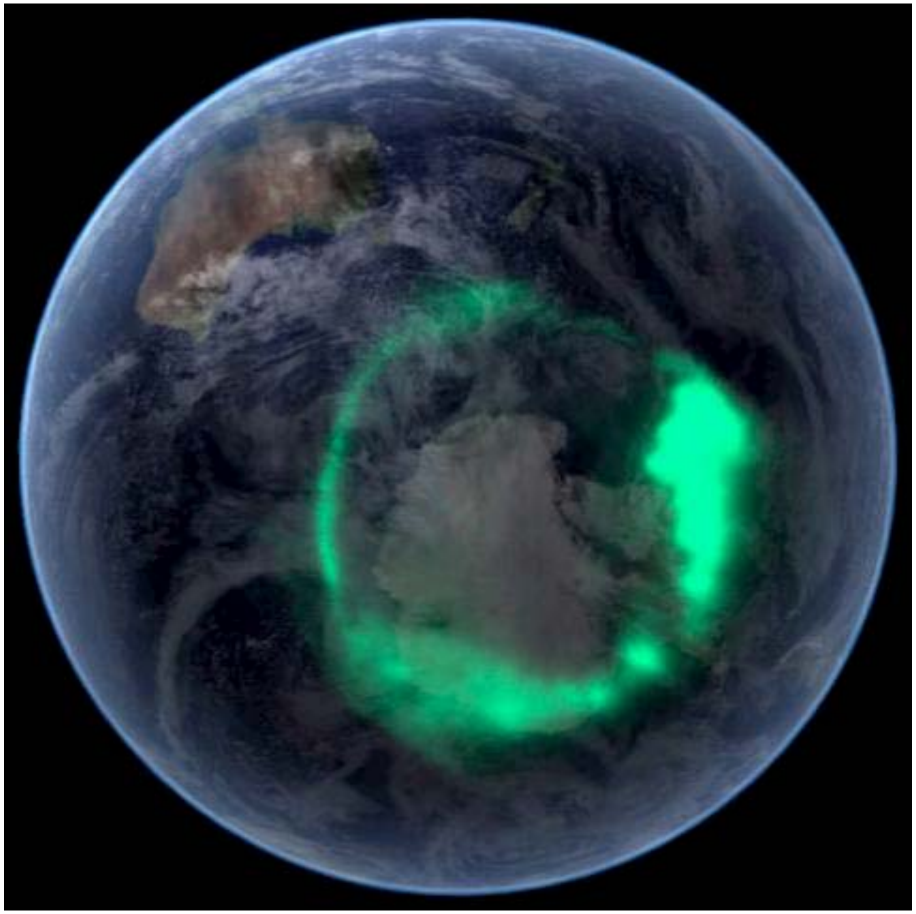
Photograph by Hey Paul on Flickr.

- **T**ransmissive $\omega < \omega_0 - \gamma/2$
- **A**bsorptive $\omega_0 - \gamma/2 < \omega < \omega_0$
- **R**eflective $\omega_0 + \gamma/2 < \omega < \omega_p$
- **T**ransmissive $\omega > \omega_p$

Plasma in Ionosphere

Plasma is an ionized gas consisting of positively charged molecules (ions) and negatively charged electrons that are free to move.

Plasma exists naturally in what we call ionosphere (80 km ~ 120 km above the surface of the Earth). Here the UV light from the Sun ionizes air molecules.



Aurora Australis

Plasmas (which we will assume to be lossless, $\gamma = 0$ *)* *... have no restoring force for electrons,* $\omega_o = 0$

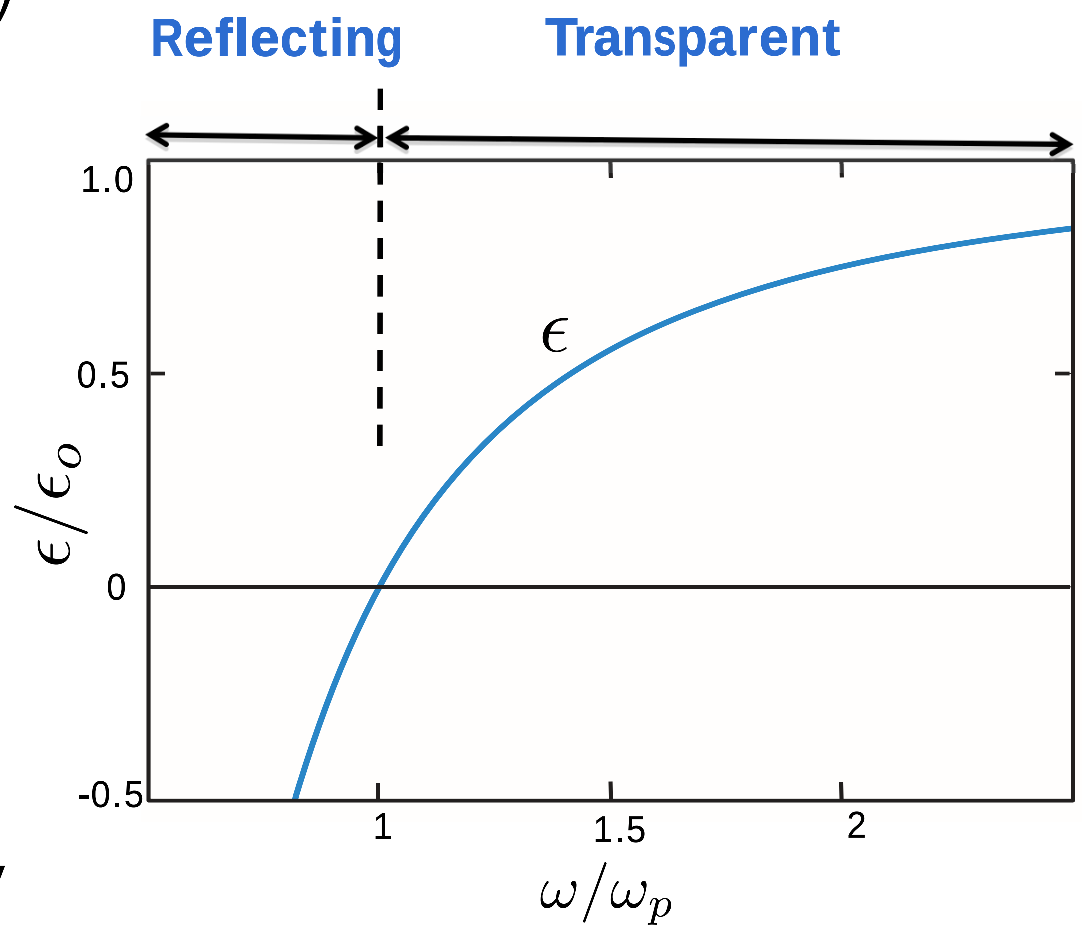
$$\epsilon = \epsilon_o \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

What happens when the dielectric constant is negative?

$\omega < \omega_p \longrightarrow \epsilon < 0$

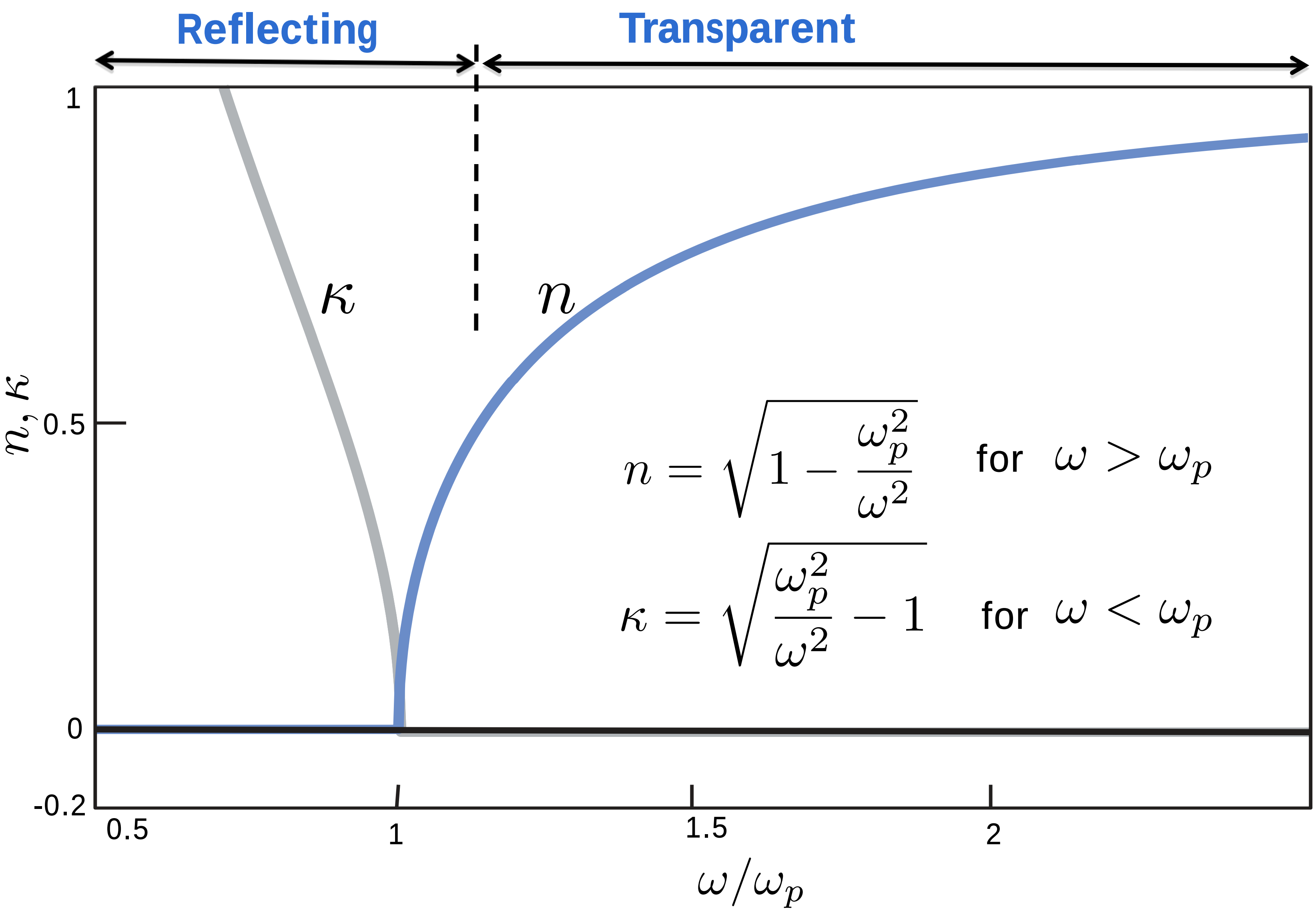
$\omega > \omega_p \longrightarrow \epsilon > 0$

If $\varepsilon < 0$ then n is imaginary



Optical Response of Plasmas

$$\sqrt{\frac{\epsilon}{\epsilon_o}} = n - j\kappa$$



Plasma Frequency

$$\omega_p = \sqrt{\frac{Ne^2}{\epsilon_o m}}$$

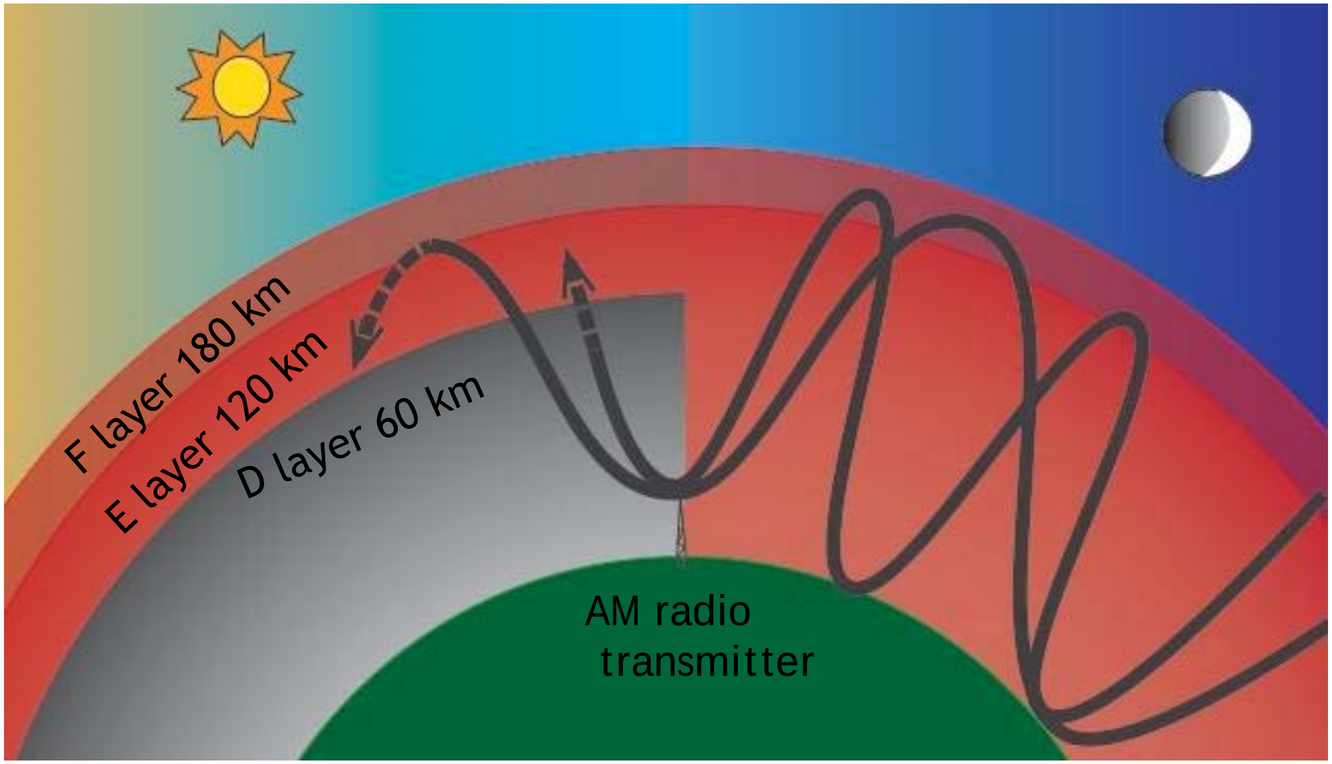
$$= \frac{(10^{12}M^{-3})(1.602x10^{-19}C)^2}{(8.854 \times 10^{-12}F/M)(9.109 \times 10^{-31}Kg)}$$

$$= 5.64 \times 10^7 rad/sec$$

$$= 2\pi \times (8.98MHz)$$

AM radio is in the range 520-1610 kHz — Reflected

FM radio in in the range 87.5 to 108 MHz — Transmitted



The Ionosphere and Radio Wave Propagation

The ionosphere is important for radio wave (AM only) propagation....

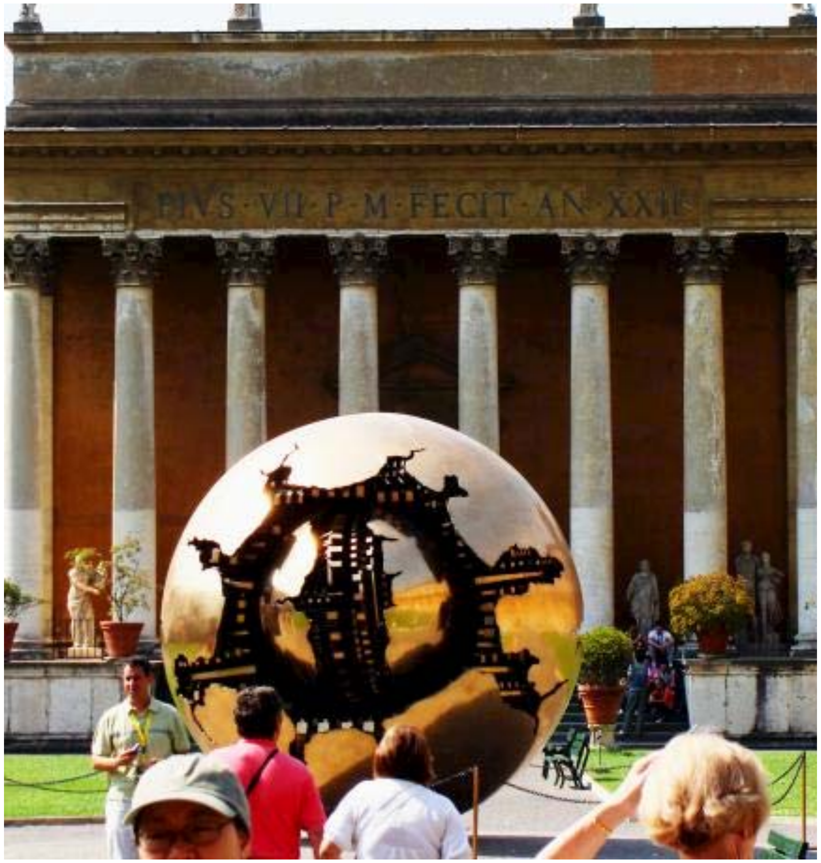
Ionosphere is composed of the D, E, and F layers.

The D layer is good at absorbing AM radio waves.

D layer disappears at night...the E and F layers bounce the waves back to the Earth.

This explains why radio stations adjust their power output at sunset and sunrise.

Why do metals reflect light?



Metals are Lossy

Metals have loss
but have no restoring force for electrons

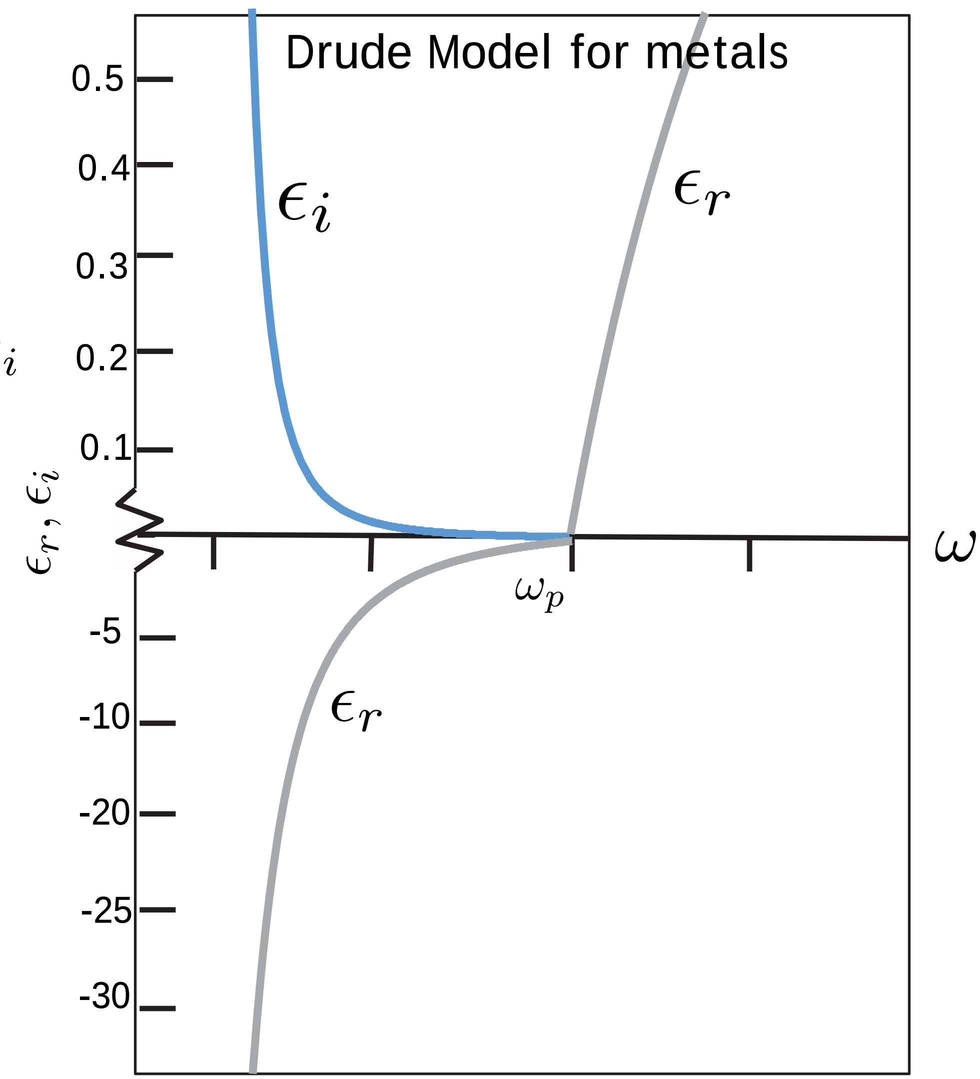
Why is there a discontinuity here?
ϵ_o or γ must change for this to be true

$$\gamma \neq 0$$
$$\omega_o = 0$$

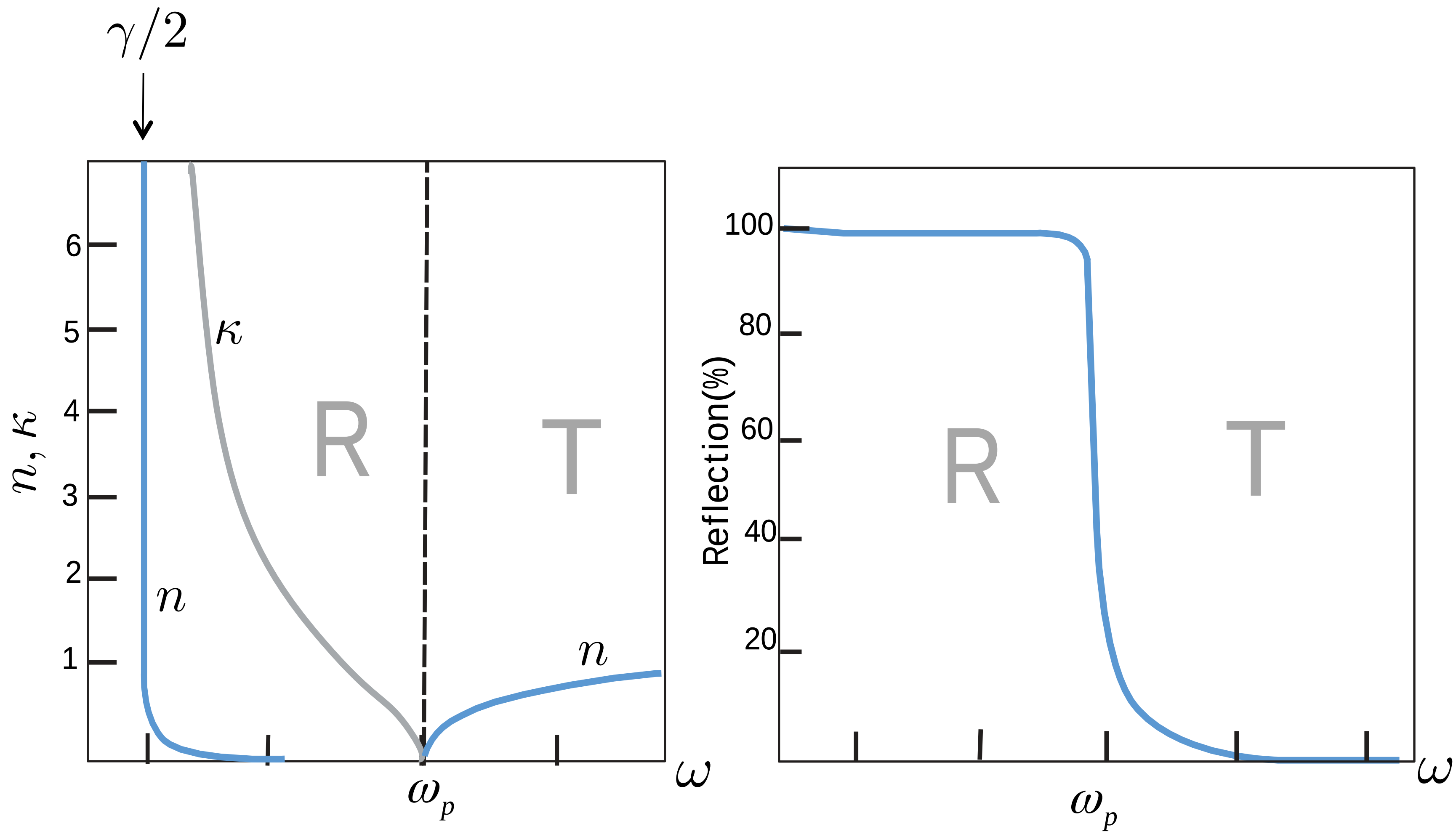
$$\epsilon = \epsilon_o \left(1 - \frac{\omega_p^2}{\omega^2 - j\omega\gamma}\right) = \epsilon_r + i\epsilon_i$$

$$\epsilon_r = \epsilon_o \left(1 - \frac{\omega_p^2}{\omega^2 + \gamma^2}\right)$$

$$\epsilon_i = \epsilon_o \left(\frac{\gamma\omega_p^2}{\omega(\omega^2 + \gamma^2)}\right)$$



Behavior of Metals

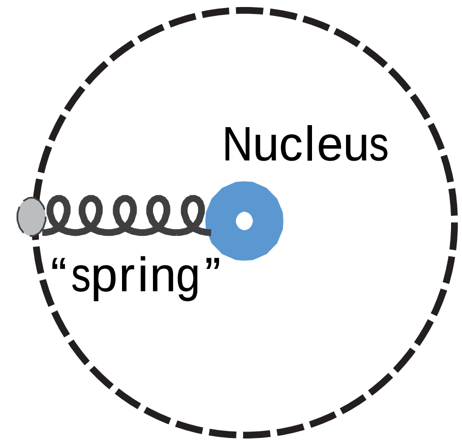


Dielectric: $\omega_0 - \gamma/2$, ω_0, $\omega_0 + \gamma/2$, ω_p

Metal: $\omega_0 = 0$, $\omega_0 + \gamma/2$, ω_p

Plasma: $\omega_0 = 0$, $\gamma = 0$, ω_p

Key Takeaways



Lorentz Oscillator Model

$$m\frac{d^2y}{dt^2} = -m\omega_o^2 y + qE_y - m\gamma\frac{dy}{dt}$$

$$\vec{P}(\omega) = \frac{\omega_p^2}{(\omega_o^2 - \omega^2) + j\omega\gamma}\epsilon_o\vec{E}(\omega) \Rightarrow \tilde{n} = n - j\kappa$$

$$\Rightarrow \epsilon = \epsilon_r - j\epsilon_i \qquad \alpha = 2k_o\kappa$$

Decay

$$E(t,z) = Re\{\tilde{E}_0 e^{-\alpha z/2} e^{j(\omega t - k_0 n z)}\}$$

Absorption coefficient

Refractive index

$$I(z) = I_o e^{-\alpha z}$$ Beer’s Law

- Transmissive $\omega < \omega_0 - \gamma/2$
- Absorptive $\omega_0 - \gamma/2 < \omega < \omega_0$
- Reflective $\omega_0 + \gamma/2 < \omega < \omega_p$
- Transmissive $\omega > \omega_p$

6.007 Electromagnetic Energy: From Motors to Lasers
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