6.013/ESD.013J — Electromagnetics and Applications	Fall 2005
Problem Set 11 - Solutions	
Prof. Markus Zahn	MIT OpenCourseWare

Problem 11.1

A

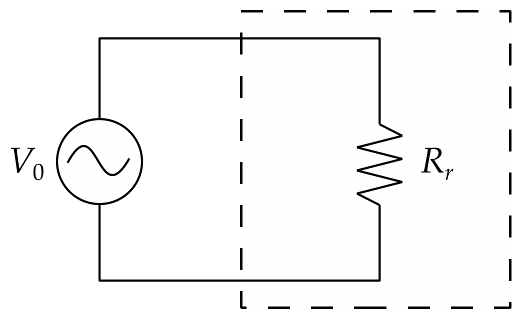


Figure 1: Impedance model. (Image by MIT OpenCourseWare.)

$$V_{\text{RMS}} = \frac{V_0}{\sqrt{2}}$$

$$P = \frac{V_{\text{RMS}}^2}{R_r} \implies V_{\text{RMS}} = \sqrt{PR_r}$$

$$\implies V_0 = \sqrt{(2)(70)(10^5)} = 3741.7 \text{ Volts (peak)}$$

B

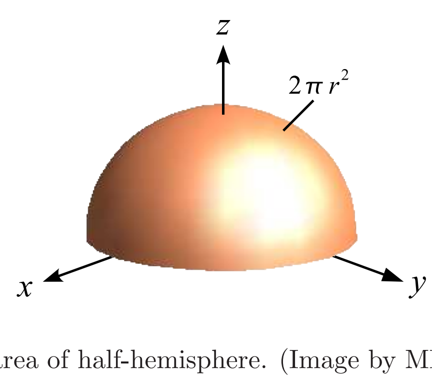


Figure 2: Surface area of half-hemisphere. (Image by MIT OpenCourseWare.)

$$2\pi r^2 I = P \implies I = \frac{P}{2\pi r^2} = \frac{100\times 10^3}{2\pi(50\times 10^3)^2} = 6.366\times 10^{-6} \text{ Watts/m}^2$$

C

$$[P_r]_{\text{max}} = IA = (6.336\times 10^{-6} \text{ W/m}^2)(10 \text{ m}^2) = 63.66\times 10^{-6} \text{ Watts}$$

Problem 11.2

A

$E(r,\theta,t)$ in the far field limit

$$\begin{aligned}\hat{\mathbf{E}} &= \text{Re}\left\{j\frac{\eta k I d}{4\pi r}e^{-jkr}\sin\theta e^{j\omega t}\,\hat{\mathbf{e}}_\theta\right\}\\ &= -\frac{\eta k I d}{4\pi r}\sin\theta\sin(\omega t - kr)\,\hat{\mathbf{e}}_\theta\end{aligned}$$

B

$$k = \frac{\omega}{c} \implies \lambda = \frac{c}{f}$$

$$P_{\text{total}} = \eta_0\frac{\pi}{3}\left|\frac{Idf}{c}\right|^2 \implies f = \frac{c}{Id}\sqrt{\frac{3P_{\text{total}}}{\pi\eta_0}}$$

$$f = \frac{3\times 10^8}{(1)(0.1)}\sqrt{\frac{3(1)}{\pi(377)}} = 150.99\times 10^6 \text{ Hz}$$

C

$$\hat{\mathbf{S}} = \hat{\mathbf{r}}\left(\frac{\eta_0}{2}\right)\left|\frac{Id}{2\lambda r}\right|^2\sin^2\theta \text{ [W/m}^2\text{]}$$

First of all, the farthest you can go is when $\theta = \pm\pi/2$ because the power directed there is maximum since $\sin^2\theta = 1$.

$$A|\mathbf{S}|_{\text{max}} = P_*$$

$$A\left(\frac{\eta_0}{2}\right)\left|\frac{Idf}{2cr}\right|^2 = P_* \implies \frac{1}{r} \approx \sqrt{\frac{2\times 10^{-20}}{(377)(0.1)}}\left(\frac{2(3\times 10^8)}{(0.1)(150.99\times 10^6)}\right)$$

$$\boxed{r \approx 1.1\times 10^9 \text{ m}}$$

Problem 11.3

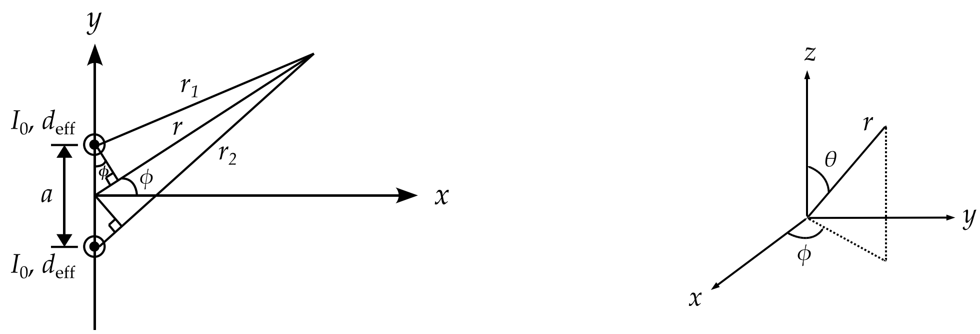


Figure 3: Dipole configuration and spherical coordinate system. (Image by MIT OpenCourseWare.)

A

Intensity of radiation in the far field? This situation is similar to that developed in lecture, but the dipoles are oriented on the y-axis rather than the x-axis.
For a single dipole, the field on the x, y-plane is

$$\hat{\mathbf{E}}(r, \theta = \frac{\pi}{2}, \phi) = \hat{\mathbf{e}}_\theta \; \eta \frac{jk\hat{I}d_{\text{eff}}}{4\pi r} e^{-jkr}$$

For two dipoles, $\hat{I}_1 = I_0$ and $\hat{I}_2 = I_0 e^{j\psi}$, both with length d_{eff}

$$\hat{E}_{\theta,\text{total}} = \eta \frac{jkI_0 d_{\text{eff}}}{4\pi r_1} e^{-jkr_1} + \eta \frac{jkI_0 d_{\text{eff}}}{4\pi r_2} e^{j\psi} e^{-jkr_2}$$

$$r_1 \approx r - \frac{a}{2}\sin\phi$$

$$r_2 \approx r + \frac{a}{2}\sin\phi$$

These small differences only matter for phase

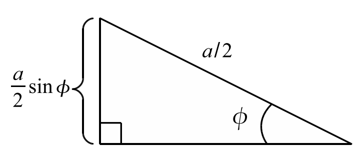


Figure 4: Triangle details. (Image by MIT OpenCourseWare.)

$$\begin{aligned}\hat{E}_{\theta,\text{total}} &= \eta\frac{jkI_0d_{\text{eff}}}{4\pi r}\left(e^{-jkr+jk\frac{a}{2}\sin\phi}+e^{-jkr-\frac{a}{2}\sin\phi}e^{j\psi}\right)\\ &= \eta\frac{jkI_0d_{\text{eff}}}{4\pi r}e^{j\psi/2}e^{-jkr}\left(e^{j(k\frac{a}{2}\sin\phi-\frac{\psi}{2})}+e^{-j(k\frac{a}{2}\sin\phi-\frac{\psi}{2})}\right)\\ &= \eta\frac{jkI_0d_{\text{eff}}}{4\pi r}e^{j\psi/2}e^{-jkr}2\cos\left(k\frac{a}{2}\sin\phi-\frac{\psi}{2}\right)\end{aligned}$$

For our case, $\psi = 0$

$$\hat{E}_{\theta,\text{total}} = \eta\frac{jkI_0d_{\text{eff}}}{4\pi r}e^{-jkr}2\cos\left(k\frac{a}{2}\sin\phi\right)$$

$$\text{Intensity} = \langle\mathbf{S}\rangle = \hat{\mathbf{e}}_r\frac{1}{2}\frac{|\hat{E}_\theta|^2}{\eta} = \hat{\mathbf{e}}_r2\eta\left(\frac{kI_0d_{\text{eff}}}{4\pi r}\right)^2\cos^2\left(k\frac{a}{2}\sin\phi\right)$$

B

$$a = 2\lambda,\ k\frac{a}{2} = \frac{2\pi}{\cancel{\lambda}}\frac{\cancel{2}\cancel{\lambda}}{\cancel{2}} = 2\pi$$

$$\langle\mathbf{S}\rangle = \hat{\mathbf{e}}_r2\eta\left(\frac{kI_0d_{\text{eff}}}{4\pi r}\right)^2\cos^2(2\pi\sin\phi)$$

ϕ_{max}?

$$\begin{aligned}&\cos^2(2\pi\sin\phi_{\text{max}}) = 1\\ &\quad 2\cancel{\pi}\sin\phi_{\text{max}} = n\cancel{\pi},\ n = 0,\pm1,\pm2,\ldots\\ &\quad\quad \sin\phi_{\text{max}} = \frac{n}{2}\end{aligned}$$

$n = 0$	$\sin\phi_{\text{max}} = 0$	$\phi_{\text{max}} = 0, 180^\circ$
$n = \pm1$	$\sin\phi_{\text{max}} = \pm1/2$	$\phi_{\text{max}} = \pm30^\circ, \pm150^\circ$
$n = \pm2$	$\sin\phi_{\text{max}} = \pm1$	$\phi_{\text{max}} = \pm90^\circ$

ϕ_{min}?

$$\begin{aligned}&\cos^2(2\pi\sin\phi_{\text{min}}) = 0\\ &\quad 2\cancel{\pi}\sin\phi_{\text{min}} = (2m-1)\frac{\cancel{\pi}}{2},\ m = 0,\pm1,\pm2,\ldots\\ &\quad\quad \sin\phi_{\text{min}} = (2m-1)\frac{1}{4}\end{aligned}$$

$m = 1, 0$	$\sin\phi_{\text{min}} = \pm1/4$	$\phi_{\text{min}} = \pm14.48^\circ, \pm165.52^\circ$
$m = 2, -1$	$\sin\phi_{\text{min}} = \pm3/4$	$\phi_{\text{min}} = \pm48.59, \pm131.41^\circ$

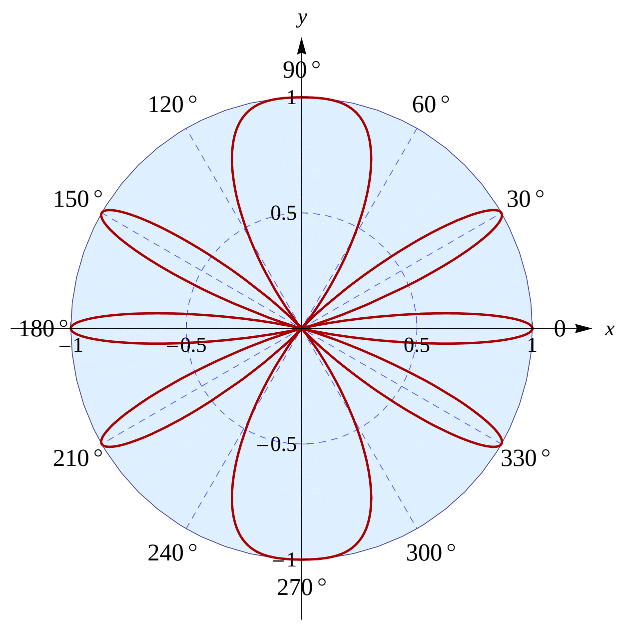


Figure 5: Plot of radiation pattern. (Image by MIT OpenCourseWare.)

Problem 11.4

A dipole in the $\hat{\mathbf{e}}_z$-direction has an electric field in the far-field, in spherical coordinates, of

$$\hat{\mathbf{E}} = \eta \frac{jkId}{4\pi r} e^{-jkr} \hat{\mathbf{e}}_\theta \sin\theta$$

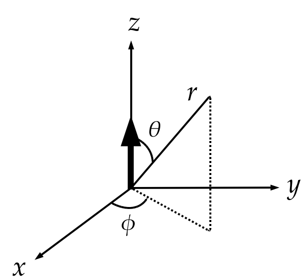


Figure 6: Dipole orientation. (Image by MIT OpenCourseWare.)

We have a dipole in the $\hat{\mathbf{e}}_x$-direction. We can rotate the cartesian system such that we can use the solution for the $\hat{\mathbf{z}}$-directed dipole. If we transform the spherical solution back to cartesian coordinates correctly we will have found our solution.

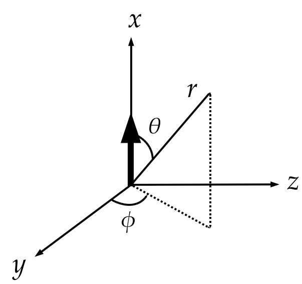


Figure 7: Dipole with rotated coordinates. (Image by MIT OpenCourseWare.)

We are only interested in the z-axis: $\theta = \pi/2,\ \phi = \pm\pi/2$

$$\hat{\mathbf{E}} = \eta \frac{jIkd}{4\pi r} e^{-jkr} \hat{\mathbf{e}}_\theta$$

On z-axis:

$$\hat{\mathbf{e}}_\theta = -\hat{\mathbf{e}}_x,\ r = |z|$$

$$\hat{\mathbf{E}} = -\hat{\mathbf{e}}_x \eta \frac{jIkd}{4\pi |z|} e^{-jk|z|}$$

This dipole has current $I = \hat{I}_0$ and length $d = d_{\text{eff}}$:

$$\hat{\mathbf{E}} = -\hat{\mathbf{e}}_x \eta \frac{j\hat{I}_0 d_{\text{eff}}}{4\pi |z|} e^{-jk|z|}$$

We also have a dipole in the $\hat{\mathbf{e}}_y$-direction. We use the same method: The z-axis: $\theta = \pi/2,\ \phi = \pm\pi/2$

$$\hat{\mathbf{E}} = \eta \frac{jIkd}{4\pi r} e^{-jkr} \hat{\mathbf{e}}_\theta$$

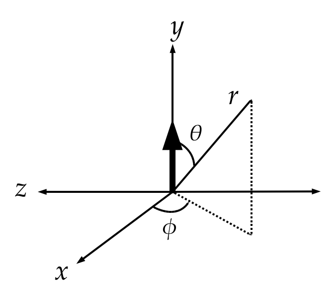


Figure 8: Dipole with rotated coordinates. (Image by MIT OpenCourseWare.)

On z-axis, $\hat{\mathbf{e}}_\theta = -\hat{\mathbf{e}}_y,\ r = |z|$

$$\hat{\mathbf{E}} = -\eta \hat{\mathbf{e}}_y \frac{jIkd}{4\pi |z|} e^{-jk|z|}$$

This dipole has $I = j\hat{I}_0$ and $d = d_{\text{eff}}$.

$$\hat{\mathbf{E}} = \eta\hat{\mathbf{e}}_y \frac{k\hat{I}_0 d_{\text{eff}}}{4\pi|z|} e^{-jk|z|}$$

Total field is given by superposition:

$$\hat{\mathbf{E}}_{\text{total}} = (-j\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y)\eta \frac{k\hat{I}_0 d_{\text{eff}}}{4\pi|z|} e^{-jk|z|}$$

On the $+z$-axis, $z > 0$

$$\hat{\mathbf{E}}_{\text{total}} = (-j\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y)\eta \frac{k\hat{I}_0 d_{\text{eff}}}{4\pi z} e^{-jkz}$$

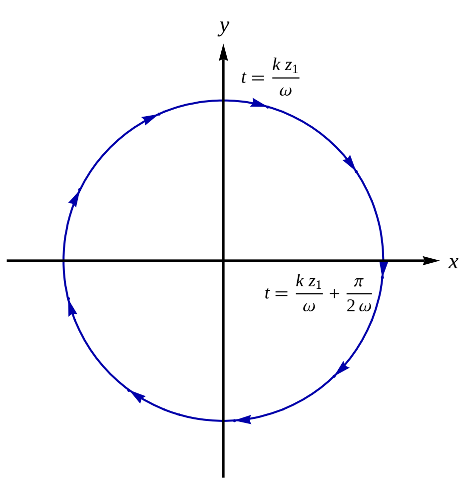
B

Polarization: As time advances, how does the direction and amplitude of the electric field change? For this, we need to look at the real E-field, not just the complex amplitude:

$$\hat{\mathbf{E}} = \text{Re}\{\hat{\mathbf{E}}e^{j\omega t}\} = \text{Re}\left\{(-j\hat{\mathbf{e}}_x[\cos(\omega t - kz) + j\sin(\omega t - kz)] + \hat{\mathbf{e}}_y[\cos(\omega t - kz) + j\sin(\omega t - kz)])\eta\frac{k\hat{I}_0 d_{\text{eff}}}{4\pi z}\right\}$$

$$\hat{\mathbf{E}} = (\hat{\mathbf{e}}_x \sin(\omega t - kz) + \hat{\mathbf{e}}_y \cos(\omega t - kz))\eta\frac{kI_0 d_{\text{eff}}}{4\pi z}$$

Let us look at one point in space, $z = z_1$, and see how the direction and magnitude of the E-field changes: Only the direction of the field changes as time advances; the magnitude remains the same. Thus, it is



$$\omega t - kz_1 = 0 \quad \rightarrow \hat{\mathbf{E}} = \hat{\mathbf{e}}_y \eta \frac{I_0 k d_{\text{eff}}}{4\pi z_1}$$
$$\omega t - kz_1 = \frac{\pi}{2} \quad \rightarrow \hat{\mathbf{E}} = \hat{\mathbf{e}}_x \eta \frac{I_0 k d_{\text{eff}}}{4\pi z_1}$$

Figure 9: Time evolution of electric field. (Image by MIT OpenCourseWare.)

circularly polarized, since the field traces out a circle.

To determine whether the polarization is right-handed or left-handed, curl your fingers of both hands in the direction of the path traced out by the field. If your right thumb points in the direction of propagation ($+z$ in this case), then the field is right-handed. If your left thumb points in the direction of propagation, however, it is left-handed. In this case we have a left-handed circularly polarized wave.

C

Find the magnetic field:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$
$$\nabla \times \hat{\mathbf{E}} = -\mu j\omega \hat{\mathbf{H}}$$

$$\begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \boxed{\frac{\partial}{\partial z}} \\ E_x & E_y & E_z \end{vmatrix} \quad \hat{\mathbf{E}} \text{ only varies with } z \implies \text{ only has } E_x, E_y \text{ components}$$

$$-\hat{\mathbf{e}}_x \frac{\partial \hat{E}_y}{\partial z} + \hat{\mathbf{e}}_y \frac{\partial \hat{E}_x}{\partial z} = -\mu j\omega \hat{\mathbf{H}}$$

We assume that $1/z$ varies much slower than e^{-jkz}, so we can treat $1/z$ as a constant:

$$\frac{\partial \hat{E}_y}{\partial z} = -jk\hat{E}_y, \qquad \frac{\partial \hat{E}_x}{\partial z} = -jk\hat{E}_x$$

$$-\hat{\mathbf{e}}_x(-jk)\eta \frac{k\hat{I}_0 d_{\text{eff}}}{4\pi z} e^{-jkz} + \hat{\mathbf{e}}_y(-jk)(-j)\eta \frac{k\hat{I}_0 d_{\text{eff}}}{4\pi z} e^{-jkz} = -\mu j\omega \hat{\mathbf{H}}$$
$$(j\hat{\mathbf{e}}_x - \hat{\mathbf{e}}_y)\eta \frac{k^2\hat{I}_0 d_{\text{eff}}}{4\pi z} e^{-jkz} = -\mu j\omega \hat{\mathbf{H}}$$

$$\hat{\mathbf{H}} = (-\hat{\mathbf{e}}_x - j\hat{\mathbf{e}}_y)\eta \frac{k^2\hat{I}_0 d_{\text{eff}}}{\mu\omega 4\pi z} e^{-jkz} = (-\hat{\mathbf{e}}_x - j\hat{\mathbf{e}}_y)\sqrt{\frac{\mu}{\varepsilon}} \frac{\omega^2 \mu\varepsilon \hat{I}_0 d_{\text{eff}}}{\mu\omega 4\pi z} e^{-jkz}$$
$$= -(\hat{\mathbf{e}}_x + j\hat{\mathbf{e}}_y) \frac{\omega\sqrt{\mu\varepsilon}\hat{I}_0 d_{\text{eff}}}{4\pi z} e^{-jkz}$$
$$= -(\hat{\mathbf{e}}_x + j\hat{\mathbf{e}}_y) \frac{k\hat{I}_0 d_{\text{eff}}}{4\pi z} e^{-jkz}$$

D

$$\langle \mathbf{S} \rangle = \frac{1}{2}\text{Re}\{\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*\}$$
$$= \frac{1}{2}\text{Re}\left\{(-j\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y)\eta \frac{kI_0 d_{\text{eff}}}{4\pi z} e^{-jkz} \times (-\hat{\mathbf{e}}_x + j\hat{\mathbf{e}}_y) \frac{kI_0 d_{\text{eff}}}{4\pi z} e^{jkz}\right\}$$
$$= \frac{1}{2}\text{Re}\left\{(\hat{\mathbf{e}}_z + \hat{\mathbf{e}}_z)\eta \left(\frac{kI_0 d_{\text{eff}}}{4\pi z}\right)^2\right\}$$
$$= \hat{\mathbf{e}}_z \eta \left(\frac{kI_0 d_{\text{eff}}}{4\pi z}\right)^2$$

Problem 11.5

A

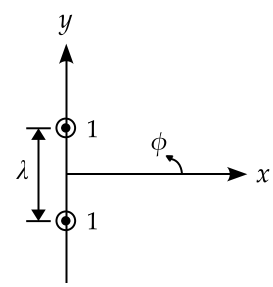


Figure 10: Dipole configuration. (Image by MIT OpenCourseWare.)

In general:

$$\begin{aligned} E_{\theta,\text{total}} &= \eta \frac{jkI_0 d_{\text{eff}}}{4\pi r_1} e^{-jkr_1} + \eta \frac{jkI_0 e^{j\psi} d_{\text{eff}}}{4\pi r_2} e^{-jkr_2} \\ &= \eta \frac{jkI_0 d_{\text{eff}}}{4\pi r} e^{j\psi/2} \left(e^{-j\psi/2} e^{jk\frac{a}{2}\sin\phi} + e^{j\psi/2} e^{-jk\frac{a}{2}\sin\phi} \right) \\ &= \eta \frac{jkI_0 d_{\text{eff}}}{4\pi r} e^{j\psi/2} 2\cos\left(k\frac{a}{2}\sin\phi - \frac{\psi}{2} \right) \end{aligned}$$

$$\langle \mathbf{S} \rangle = \hat{\mathbf{e}}_r \frac{1}{2} \frac{|\hat{E}_\theta|^2}{\eta} = \hat{\mathbf{e}}_r 2\eta \left(\frac{kI_0 d_{\text{eff}}}{4\pi r} \right)^2 \cos^2\left(k\frac{a}{2}\sin\phi - \frac{\psi}{2} \right)$$

$$a = \lambda,\ k\frac{a}{2} = \frac{\cancel{2}\pi}{\cancel{\lambda}} \frac{\cancel{\lambda}}{\cancel{2}} = \pi,\ \psi = 0$$

$$\langle \mathbf{S} \rangle = \hat{\mathbf{e}}_r 2\eta \left(\frac{kI_0 d_{\text{eff}}}{4\pi r} \right)^2 \cos^2(\pi \sin\phi)$$

Nulls:

$$\cos(\pi\sin\phi) = 0$$

$$\cancel{\pi}\sin\phi = (2n-1)\frac{\cancel{\pi}}{2},\ n = 0, \pm 1, \pm 2, \ldots$$

$$\sin\phi = (2n-1)\frac{1}{2}$$

$$n = 1, 0 \quad \sin\phi = \pm\frac{1}{2} \quad \phi = \pm 30^\circ, \pm 150^\circ$$

Peaks:

$$\cos^2(\pi\sin\phi) = 1$$

$$\cancel{\pi}\sin\phi = m\cancel{\pi},\ m = 0, \pm 1, \pm 2, \ldots$$

$$\sin\phi = m$$

$$m = 0 \quad \sin\phi = 0 \quad \phi = 0, 180^\circ$$
$$m = \pm 1 \quad \sin\phi = \pm 1 \quad \phi = \pm 90^\circ$$

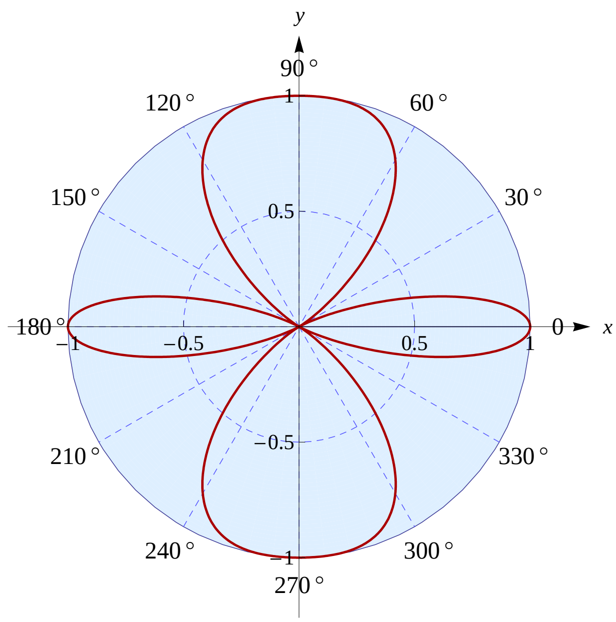


Figure 11: Plot of radiation pattern. (Image by MIT OpenCourseWare.)

B

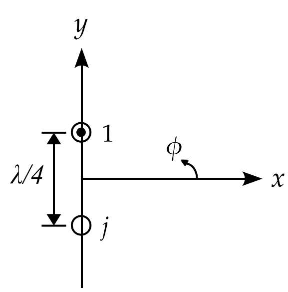


Figure 12: Dipole configuration. (Image by MIT OpenCourseWare.)

$$I_2 = I_1 e^{j\psi} \quad \psi = \frac{\pi}{2} \quad a = \frac{\lambda}{4}$$

$$\frac{ka}{2} = \frac{\pi}{4}$$

$$\langle \mathbf{S} \rangle = \hat{\mathbf{e}}_r 2\eta \left(\frac{kI_0 d_{\text{eff}}}{4\pi r} \right)^2 \cos^2 \left(\frac{\pi}{4} \sin\phi - \frac{\pi}{4} \right)$$

Nulls:

$$\cos\left(\frac{\pi}{4}\sin\phi - \frac{\pi}{4}\right) = 0$$

$$\frac{\cancel{\pi}}{4}\sin\phi - \frac{\cancel{\pi}}{4} = (2m-1)\frac{\cancel{\pi}}{2},\ m = 0, \pm 1, \pm 2, \ldots$$

$$\sin\phi = 4m - 1$$

$m = 0 \quad \sin\phi = -1 \quad \phi = -90^\circ$

Peaks:

$$\cos^2\left(\frac{\pi}{4}\sin\phi - \frac{\pi}{4}\right) = 1$$

$$\frac{\cancel{\pi}}{4}\sin\phi - \frac{\cancel{\pi}}{4} = (2m-1)\frac{\cancel{\pi}}{2},\ m = 0, \pm 1, \pm 2, \ldots$$

$$\sin\phi = 4m + 1$$

$n = 0 \quad \sin\phi = 1 \quad \phi = 90^\circ$

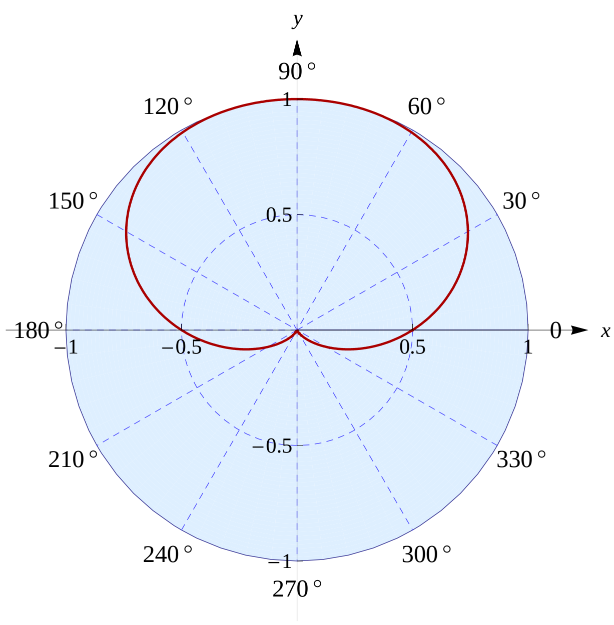


Figure 13: Plot of radiation pattern. (Image by MIT OpenCourseWare.)

C

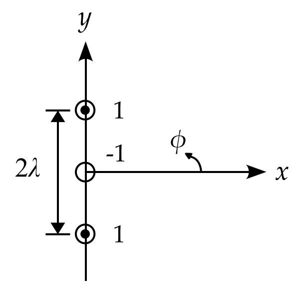


Figure 14: Dipole configuration. (Image by MIT OpenCourseWare.)

$$\hat{E}_\theta(r, \theta = \frac{\pi}{2}, \phi) = \frac{jk\eta d_{\text{eff}}}{4\pi r} \underbrace{\left[\sum_{-N}^{N} \hat{I}_N e^{jkn\frac{a}{2}\sin\phi}\right]}_{\text{array factor}} e^{-jkr}, \qquad I_2 = -I_1, I_3 = I_1$$

$$\begin{aligned}\hat{E}_\theta(r, \theta = \frac{\pi}{2}, \phi) &= \frac{jk\eta d_{\text{eff}}}{4\pi r} I_0 \left[e^{jk\frac{a}{2}\sin\phi} - 1 + e^{-jk\frac{a}{2}\sin\phi}\right] e^{-jkr} \\ &= \frac{jk\eta d_{\text{eff}} I_0}{4\pi r} e^{-jkr}[2\cos\left(k\frac{a}{2}\sin\phi\right) - 1]\end{aligned}$$

$$\langle \mathbf{S} \rangle = \hat{\mathbf{e}}_r \eta \left(\frac{kI_0 d_{\text{eff}}}{4\pi r}\right)^2 \left[2\cos(k\frac{a}{2}\sin\phi) - 1\right]^2$$

$$a = 2\lambda, k\frac{a}{2} = \frac{2\pi}{\cancel{\lambda}}\frac{\cancel{2}\cancel{\lambda}}{\cancel{2}} = 2\pi$$

$$\langle \mathbf{S} \rangle = \hat{\mathbf{e}}_r \eta \left(\frac{kI_0 d_{\text{eff}}}{4\pi r}\right)^2 [2\cos(2\pi\sin\phi) - 1]^2$$

Nulls:

$$\begin{aligned}2\cos(2\pi\sin\phi) &= 1 \\ \cos(2\pi\sin\phi) &= \frac{1}{2} \\ 2\pi\sin\phi &= \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}, \pm\frac{7\pi}{3}, \ldots \\ \sin\phi &= \pm\frac{1}{6}, \pm\frac{5}{6}, \cancel{\pm\frac{7}{6}} \text{ (larger than 1)}\end{aligned}$$

$$\phi = \pm 9.59^\circ, \pm 170.41^\circ, \pm 62.71^\circ, \pm 117.29^\circ$$

Peaks:

Largest Peaks?

$$\cos(2\pi \sin\phi) = -1 \qquad [2\cos(2\pi\sin\phi) - 1]^2 = 9$$
$$2\pi \sin\phi = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$$
$$\sin\phi = \pm\frac{1}{2}, \cancel{\pm\frac{3}{2}} \implies \phi = \pm 30^\circ, \pm 150^\circ$$

Smaller Peaks?

$$\cos(2\pi\sin\phi) = 1 \qquad [2\cos(2\pi\sin\phi) - 1]^2 = 1$$
$$2\pi\sin\phi = 0, \pm 2\pi, \ldots$$
$$\sin\phi = 0, \pm 1$$
$$\phi = 0^\circ, 180^\circ, \pm 90^\circ$$

What about $\cos(2\pi\sin\phi) = 0$? Though $[2\cos(2\pi\sin\phi) - 1]^2 = 1$ as well, when $\cos(2\pi\sin\phi) = 0$, this is not a peak as can be seen by taking the second derivative with respect to ϕ and evaluating it at that point.

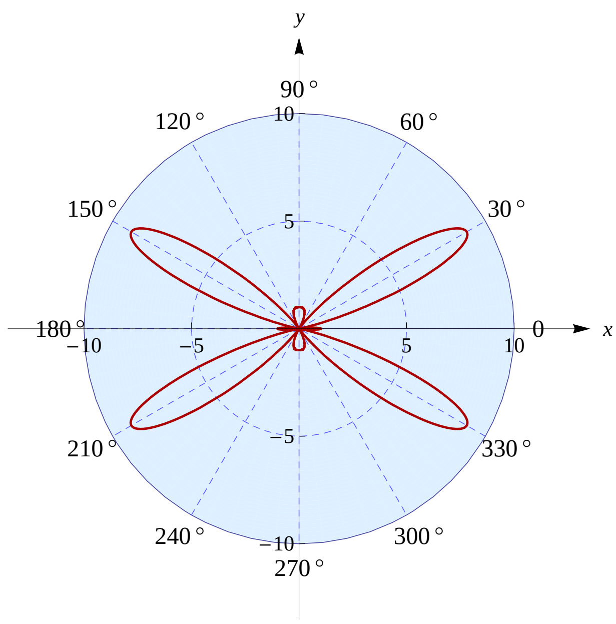


Figure 15: Plot of radiation pattern. (Image by MIT OpenCourseWare.)

Problem 11.6

A

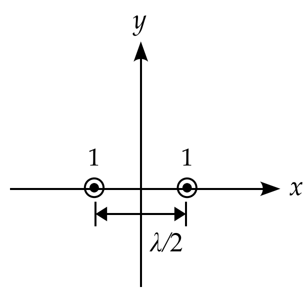


Figure 16: Dipole configuration. (Image by MIT OpenCourseWare.)

Putting 2 identical dipoles 1/2 a wavelength apart means they will cancel along the x-axis. But since neither is delayed with respect to each other, they add on the y-axis to a maximum.

$$\langle \mathbf{S} \rangle = \hat{\mathbf{e}}_r 2\eta \left(\frac{kI_0 d_{\text{eff}}}{4\pi r} \right)^2 \cos^2 \left(\frac{\pi}{2} \cos\phi \right)$$

B

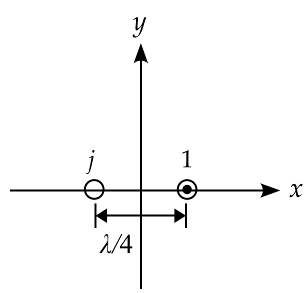


Figure 17: Dipole configuration. (Image by MIT OpenCourseWare.)

This is the same pattern as in 11.3(b), but rotated.

$$\langle \mathbf{S} \rangle = \hat{\mathbf{e}}_r 2\eta \left(\frac{kI_0 d_{\text{eff}}}{4\pi r} \right)^2 \cos^2 \left(\frac{\pi}{4} \cos\phi - \frac{\pi}{4} \right)$$

C

We need to come up with a maximum at $\phi = 0$, but a minimum at $\phi = \pi$. We have 2 dipoles of equal amplitude, separated by a distance a.

$$E = \eta \frac{jkI_1 d_{\text{eff}}}{4\pi r} e^{j\psi/2} e^{-jkr} 2\cos\left(k\frac{a}{2}\cos\phi - \frac{\psi}{2}\right)$$

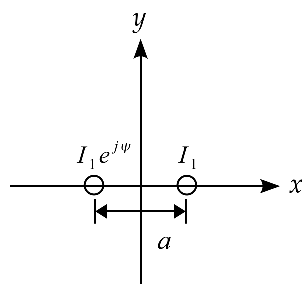


Figure 18: Dipole configuration. (Image by MIT OpenCourseWare.)

$\phi = 0 \implies$ must add to a peak

$$2\cos\left(k\frac{a}{2} - \frac{\psi}{2}\right) = 2$$

$$k\frac{a}{2} - \frac{\psi}{2} = 0, \ \pm 2\pi, \pm 4\pi, \ldots$$

$\phi = \pi \implies$ must be a null

$$-k\frac{a}{\cancel{2}} - \frac{\psi}{\cancel{2}} = \pm\frac{\pi}{\cancel{2}}, \pm\frac{3\pi}{\cancel{2}}, \ldots$$

We want the solution with the fewest nulls and peaks, so let us take the lowest angles:

$$\begin{aligned} ka - \psi &= 0 \\ + \ -ka - \psi &= \pi \\ \hline -2\psi &= \pi \\ \implies \psi &= -\tfrac{\pi}{2} \end{aligned} \implies$$

We need a positive a, so we use

$$\psi = \tfrac{3\pi}{2} = -\tfrac{\pi}{2}$$
$$ka - \tfrac{3\pi}{2} = 0$$
$$a = \tfrac{3\pi}{2}\left(\tfrac{\lambda}{2\pi}\right) = \tfrac{3\lambda}{4}$$

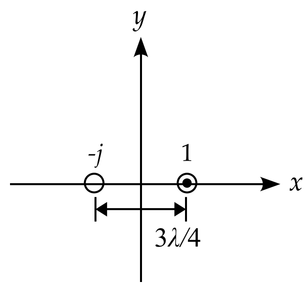


Figure 19: Dipole configuration. (Image by MIT OpenCourseWare.)

$$\langle \mathbf{S} \rangle = \hat{\mathbf{e}}_r 2\eta \left(\frac{kI_0 d_{\text{eff}}}{4\pi r} \right)^2 \cos^2 \left(\frac{3\pi}{4} \cos\phi + \frac{\pi}{4} \right)$$