

Chapter 1: Introduction to Electromagnetics and Electromagnetic Fields

1.1 *Review of foundations*

1.1.1 Introduction

Electromagnetics involves the macroscopic behavior of electric charges in vacuum and matter. This behavior can be accurately characterized by the Lorentz force law and Maxwell's equations, which were derived from experiments showing how forces on charges depend on the relative locations and motions of other charges nearby. Additional relevant laws of physics include Newton's law, photon quantization, and the conservation relations for charge, energy, power, and momentum. Electromagnetic phenomena underlie most of the “electrical” in “electrical engineering” and are basic to a sound understanding of that discipline.

Electrical engineering has delivered four “miracles” — sets of phenomena that could each be considered true magic prior to their development. The first of these to impress humanity was the electrical phenomenon of lightning, often believed to be a tool of heaven, and the less powerful magnetic force that caused lodestones to point north. The explanation and application of these invisible forces during the eighteenth and nineteenth centuries vaulted electrical engineering to the forefront of commercial interest as motors, generators, electric lights, batteries, heaters, telephones, record players, and many other devices emerged.

The second set of miracles delivered the ability to communicate instantly without wires around the world, not only dots and dashes, but also voice, images, and data. Such capabilities had been commonplace in fairy tales, but were beyond human reach until Hertz demonstrated radiowave transmission in 1888, 15 years after Maxwell's predictions. Marconi extended the technique to intercontinental distances.

Third came electronics and photonics — the ability to electrically manipulate individual electrons and atoms in vacuum and in matter so as to generate, amplify, manipulate, and detect electromagnetic signals. During the twentieth century vacuum tubes, diodes, transistors, integrated circuits, lasers, and superconductors all vastly extended the capabilities and applications of electromagnetics.

The fourth set of electrical phenomena involves cybernetics and informatics — the manipulation of electrical signals so complex that entirely new classes of functionality are obtained, such as optimum signal processing, computers, robotics, and artificial intelligence. This text focuses on the electromagnetic nature of the first three sets of phenomena and explores many of their most important applications.

Chapter 1 of this text begins with a brief review of the underlying laws of physics, followed by the Lorentz force law and the nature of electric and magnetic fields. Chapter 2 introduces electrostatics and Maxwell's equations, leading to uniform plane waves in space and media, and definitions of power, energy, boundary conditions, and uniqueness. The next four chapters

address static and quasistatic systems beginning with Chapter 3, which explores electromagnetics in the context of RLC circuits and devices. Chapter 4 addresses the more general behavior of quasistatic electric and magnetic fields in homogeneous and inhomogeneous media. Chapter 5 introduces electromagnetic forces while Chapter 6 addresses their application to motors, generators, actuators, and sensors.

The second half of the text focuses on electrodynamics and waves, beginning with TEM transmission lines in Chapters 7 and 8, and waves in media and at boundaries in Chapter 9. Antennas and radiation are treated in Chapters 10 and 11, while optical and acoustic systems are addressed in Chapters 12 and 13, respectively. Acoustics is introduced on its own merits and as a useful way to review electromagnetic wave phenomena such as radiation and resonance in a more physical and familiar context. The appendices list natural constants and review some of the prerequisite mathematics.

The rationalized international system of units (rationalized SI units) is used, which largely avoids factors of 4π . SI units emphasize meters (m), kilograms (kg), seconds (s), Amperes (A), and Kelvins (K); most other units can be expressed in terms of these. The SI system also favors units in multiples of 10^3 ; for example, it favors meters and millimeters over centimeters. The algebraic convention used here is that operations within parentheses are performed before others. Within parentheses and exponents and elsewhere, exponentiation is performed first, and multiplication before division; all these operations are performed before addition and subtraction.

1.1.2 Review of basic physical concepts and definitions

The few basic concepts summarized below are central to electromagnetics. These concepts include conservation of energy, power, and charge, and the notion of a photon, which conveys one quantum of electromagnetic energy. In addition, Newton's laws characterize the kinematics of charged particles and objects influenced by electromagnetic fields. The conservation laws also follow from Maxwell's equations, which are presented in Section 2.1 and, together with the Lorentz force law, compress all macroscopic electromagnetic behavior into a few concise statements.

This text neglects relativistic issues introduced when mass approaches the velocity of light or is converted to or from energy, and therefore we have *conservation of mass*: the total mass m within a closed envelope remains constant.

Conservation of energy requires that the total energy w_T [Joules] remains constant within any system closed so that no power enters or leaves, even though the form of the internally stored energy may be changing. This total energy w_T may include electric energy w_e , magnetic energy w_m , thermal energy w_{Th} , mechanical kinetic energy w_k , mechanical potential energy w_p , and energy in chemical, atomic, or other forms w_{other} ; w_{other} is neglected here. Conservation of energy means:

$$w_T = w_e + w_m + w_k + w_p + w_{Th} + w_{other} \text{ [Joules]} = \text{constant} \quad (1.1.1)$$

In this text we generally use lower case letters to indicate totals, and upper case letters to indicate densities. Thus we represent total energy by w_T [J] and total energy density by W_T [J m⁻³]. Similarly, f [N] denotes the total force on an object and F [N m⁻³] denotes the force density.

Unfortunately the number of electromagnetic variables is so large that many letters are used in multiple ways, and sometimes the meaning must be extracted from the context. For example, the symbol f is used to signify both force and frequency.

Newton's law says that a one-Newton force f would cause an otherwise force-free kilogram mass to accelerate at one meter per second per second; this defines the *Newton*. One Newton is roughly the terrestrial gravitational force on a quarter-pound weight (e.g. the weight of the apple that allegedly fell on Newton's head, prompting him to conceive the law of gravity). Newton's law may be expressed as:

$$f = ma \text{ [Newtons]} \quad (1.1.2)$$

where m is the mass of the object [kg] and a is the induced acceleration [ms⁻²].

The unit of energy, the *Joule*, is the total energy w_T delivered to an object when a force f of one Newton is applied to it as it moves one meter in the direction z of the force. Therefore:

$$f = \frac{dw_T}{dz} \quad (1.1.3)$$

The *kinetic energy* w_k of a mass m moving at velocity v is:

$$w_k = \frac{1}{2}mv^2 \text{ [J]} \quad (1.1.4)$$

which, when added to its *potential energy* w_p , equals its total energy w_T relative to a motionless reference position; i.e.:

$$w_T = w_k + w_p \quad (1.1.5)$$

It is easy to see that if w_p remains constant, then (1.1.3) and (1.1.4) are consistent with $f = ma$; that is, $f = dw_T/dz = dw_k/dz = mv \, dv/dz = m(dz/dt)(dv/dz) = m \, dv/dt = ma$.

Conservation of power means, for example, that the total power P_{in} [Js⁻¹] entering a closed volume must equal the rate of increase [Js⁻¹] of the total energy stored there; that is:

$$P_{in} \text{ [W]} = \frac{dw_T}{dt} \text{ [Js}^{-1}\text{]} \quad (1.1.6)$$

where dw_T/dt is the time derivative of w_T , and the units [Joules per second] are often replaced by their equivalent, Watts [W]. If $dw_T/dt = 0$, then the power flowing into a closed volume must equal the power flowing out so that power is conserved. These laws also apply to

electromagnetic power and energy, and their definition in terms of electromagnetic fields appears in Section 2.7.

In mechanical systems one watt is delivered to an object if it received one joule in one second. More generally the *mechanical power* P delivered to an object is $P = f v$ [W], where f is the only force [N] acting on the object, and v [ms⁻¹] is the object's velocity in the same direction as the force vector \vec{f} . More generally,

$$P = \vec{f} \bullet \vec{v} \equiv f v \cos \theta \text{ [W]} \quad (1.1.7)$$

where \vec{v} is the velocity vector and θ is the angle between \vec{f} and \vec{v} .

Conservation of momentum requires that the total momentum of a set of interacting masses m_i remains constant if the set is free from external forces. The *momentum* of any object is $m v$ [kg ms⁻¹], so in a force-free environment:

$$d\left(\sum_i m_i v_i\right)/dt = 0 \quad (1.1.8)$$

Conservation of charge requires that the total electric charge Q inside any volume must remain constant if no net charge crosses the boundaries of that volume. This is analogous to *conservation of mass*, although nuclear and other processes can convert mass m to energy E and vice-versa ($E = mc^2$). Charge conservation, however, has no significant exceptions. *Electric charge* is generally quantized in positive or negative multiples of the charge e on an *electron*, where:

$$e = - 1.6021 \times 10^{-19} \text{ Coulombs} \quad (1.1.9)$$

The unit of charge, one *Coulomb*, is the charge conveyed by one Ampere flowing for one second, where the *Ampere* is the unit of electric current.

Photons carry the smallest unit of energy that can be conveyed by electromagnetic waves. The energy E of a single photon is:

$$E = hf \text{ [J]} \quad (1.1.10)$$

where h is *Planck's constant* (6.624×10^{-34} [J s]) and f is the photon frequency [Hz]. Sometimes it is more convenient to think of electromagnetic waves as continuous waves, and sometimes it is more convenient to think of them as consisting of particles (photons), each of energy E. The total power P conveyed by an electromagnetic wave at frequency f is therefore the number N of photons passing per second times the photon energy E:

$$P = N hf \text{ [W]} \quad (1.1.11)$$

The frequency of a wave is simply related to its wavelength λ and the velocity of light c:

$$f = c/\lambda \quad (1.1.12)$$

Example 1.1A

A typical fully charged 1-kilowatt-hour car battery can accelerate a perfectly efficient 1000-kg electric automobile to what maximum speed?

Solution: The battery energy w_e [J] equals 1000 watts times 3600 seconds (one kilowatt-hour). It also equals the maximum kinetic energy, $w_k = mv^2/2$, of the speeding automobile (mass = $m = 1000$, velocity = v) after the battery is totally drained. Therefore $w_k = 3.6 \times 10^6 \Rightarrow v = (2w_k/m)^{0.5} = (7.2 \times 10^6/1000)^{0.5} \cong 85 \text{ m s}^{-1} \cong 190 \text{ mph}$.

Example 1.1B

A sunny day delivers $\sim 1 \text{ kw m}^{-2}$; to how many photons N per second per square meter does this correspond if we (incorrectly) assume they all have the same wavelength $\lambda = 5 \times 10^{-7}$ meters? (0.5 microns is in the visible band.)?

Solution: Power = $Nhf = Nhc/\lambda = 1 \text{ kw}$, so $N = 10^3\lambda/hc \cong 10^3 \times 5 \times 10^{-7} / (6.6 \times 10^{-34} \times 3 \times 10^8) \cong 2.5 \times 10^{20} \text{ photons m}^{-2}\text{s}^{-1}$.

1.2 Forces and the measurement and nature of electromagnetic fields

Electric fields \vec{E} and magnetic fields \vec{H} are manifest only by the forces they exert on free or bound electric charges q [Coulombs]. These forces are completely characterized by the *Lorentz force law*:

$$\vec{f}[\text{N}] = q(\vec{E} + \vec{v} \times \mu_0 \vec{H}) \quad (\text{Lorentz force law}) \quad (1.2.1)$$

Thus we can define *electric field* \vec{E} (volts/meter) in terms of the observable force vector \vec{f} :

$$\vec{E}[\text{v/m}] = \vec{f}/q \quad (\text{electric field}) \quad (1.2.2)$$

for the special case of a charge q with velocity $\vec{v} = 0$.

Similarly we can define *magnetic field* \vec{H} [A m^{-1}] in terms of the observed force vector \vec{f} given by the Lorentz force equation when $\vec{E} = 0$; \vec{H} can be sensed only by charges in motion relative to the observer. Although a single measurement of force on a motionless charge suffices to determine \vec{E} , measurements of two charge velocity vectors \vec{v} or current directions \vec{I} are required to determine \vec{H} . For example, the arbitrary test charge velocity vector $\hat{x} v_1$ yields $f_1 = q\mu_0 v_1 H \sin\theta$, where θ is the angle between \vec{v} and \vec{H} in the \hat{x} - \hat{y} plane (see Figure 1.2.1). The unit vector \hat{y} is defined as being in the observed direction $\vec{f}_1 \times \vec{v}_1$ where \vec{f}_1 defines the \hat{z} axis. A second measurement with the test charge velocity vector $\hat{y} v_2$ yields $f_2 = q\mu_0 v_2 H \cos\theta$. If $v_1 =$

v_2 then the force ratio $f_1/f_2 = \tan \theta$, yielding θ within the \hat{x} - \hat{y} plane, plus the value of H : $H = f_1/(q\mu_0 v_1 \sin \theta)$. There is no other physical method for detecting or measuring static electric or magnetic fields; we can only measure the forces on charges or on charged bodies, or measure the consequences of that force, e.g., by measuring the resulting currents.

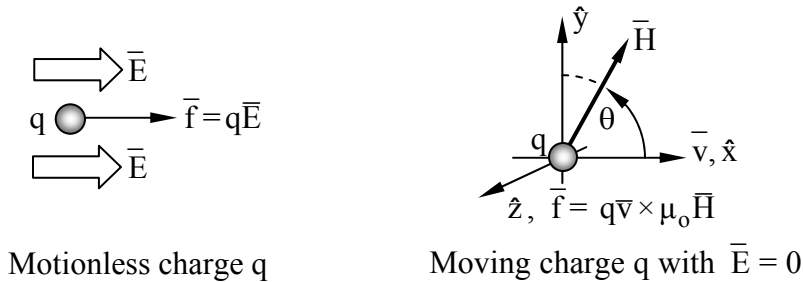


Figure 1.2.1 Measurement of electric and magnetic fields using charges.

It is helpful to have a simple physical picture of how fields behave so that their form and behavior can be guessed or approximately understood without recourse to mathematical solutions. Such physical pictures can be useful even if they are completely unrelated to reality, provided that they predict all observations in a simple way. Since the Lorentz force law plus Maxwell's equations explain essentially all non-relativistic and non-quantum electromagnetic behavior in a simple way using the fields \bar{E} and \bar{H} , we need only to ascertain how \bar{E} and \bar{H} behave given a particular distribution of stationary or moving charges q .

First consider static distributions of charge. Electric field lines are parallel to \bar{E} , and the strength of \bar{E} is proportional to the density of those field lines. Electric field lines begin on positive charges and terminate on negative ones, and the more charge there is, the more field lines there are. Field strength is proportional to lines per square meter. These lines pull on those charges to which they are attached, whether positive or negative, much as would a rubber band. Like rubber bands, they would also like to take the shortest path between two points, except that they also tend to repel their neighbors laterally, as do the charges to which they are attached.

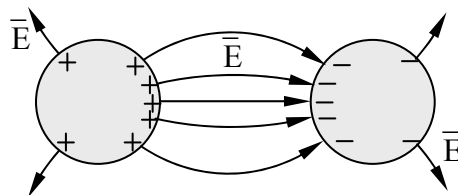


Figure 1.2.2 Electric field lines between two conducting cylinders.

Figure 1.2.2 illustrates the results of this mutual field-line repulsion, even as they pull opposite charges on conducting cylinders toward one another. Later we shall see that such electric field

lines are always perpendicular to perfectly conducting surfaces. Although these lines are illustrated as discrete, they actually are a continuum, even if only two charges are involved.

The same intuition applies to magnetic field lines \vec{H} . For example, Figure 1.2.2 would apply if the two cylinders corresponded instead to the north (+) and south (-) poles of a magnet, and if \vec{E} became \vec{H} , although \vec{H} need not emerge perfectly perpendicular to the magnet surface. In this case too the field lines would physically pull the two magnet poles toward one another. Both electric and magnetic motors can be driven using either the attractive force along field lines or the lateral repulsive force between lines, depending on motor design, as discussed later.

Another intuitive picture applies to time-dependent electromagnetic waves, where distributions of position-dependent electric and magnetic fields at right angles propagate as plane waves in the direction $\vec{E} \times \vec{H}$ much like a rigid body at the speed of light c , $\sim 3 \times 10^8$ m/s. Because electromagnetic waves can superimpose, it can be shown that any distribution of electric and magnetic fields can be considered merely as the superposition of such plane waves. Such plane waves are introduced in Section 2.2. If we examine such superpositions on spatial scales small compared to a wavelength, both the electric and magnetic fields behave much as they would in the static case.

Example 1.2A

A typical old vacuum tube accelerates electrons in a $\sim 10^4$ v m⁻¹ electric field. What is the resulting electron velocity $v(t)$ if it starts from rest? How long (τ) does it take the electron to transit the 1-cm tube?

Solution: Force $f = ma = qE$, and so $v = at = qEt/m \cong 1.6 \times 10^{-19} \times 10^4 t / (9.1 \times 10^{-28}) \cong 1.8 \times 10^{12} t$ [m s⁻¹]. Obviously v cannot exceed the speed of light c , $\sim 3 \times 10^8$ m/s. In this text we deal only with non-relativistic electrons traveling much slower than c . Distance traveled = $d = a\tau^2/2 = 0.01$, so the transit time $\tau = (2d/a)^{0.5} = (2dm/qE)^{0.5} \cong [2 \times 0.01 \times 9.1 \times 10^{-28} / (1.6 \times 10^{-19} \times 10^4)]^{0.5} \cong 1.1 \times 10^{-7}$ seconds. This slow transit limited most vacuum tubes to signal frequencies below several megahertz, although smaller gaps and higher voltages have enabled simple tubes to reach 100 MHz and higher. The microscopic gaps of semiconductors can eliminate transit time as an issue for most applications below 1 GHz; other phenomena often determine the frequency range instead.

1.3 Gauss's Law and electrostatic fields and potentials

While the Lorentz force law defines how electric and magnetic fields can be observed, Maxwell's four equations explain how these fields can be created directly from charges and currents, or indirectly and equivalently from other time varying fields. One of those four equations is *Gauss's Law for charge*, which states that the total charge Q [Coulombs] within volume V equals the integral of the normal component of the *electric displacement vector* \vec{D} over the surface area A of that volume:

$$\oiint_A (\bar{D} \cdot \hat{n}) da = \iiint_V \rho dv = Q \quad (\text{Gauss's Law for charge}) \quad (1.3.1)$$

In vacuum:

$$\bar{D} = \epsilon_0 \bar{E} \quad (1.3.2)$$

where the permittivity of vacuum $\epsilon_0 = 8.854 \times 10^{-12}$ Farads/m. Equation (1.3.1) reveals the dimensions of \bar{D} : Coulombs/m², often abbreviated here as [C/m²].

A few simple examples illustrate typical electric fields for common charge distributions, and how Gauss's law can be used to compute those fields. First consider a sphere of radius R uniformly filled with charge of density ρ_0 [C/m³], as illustrated in Figure 1.3.1(a).

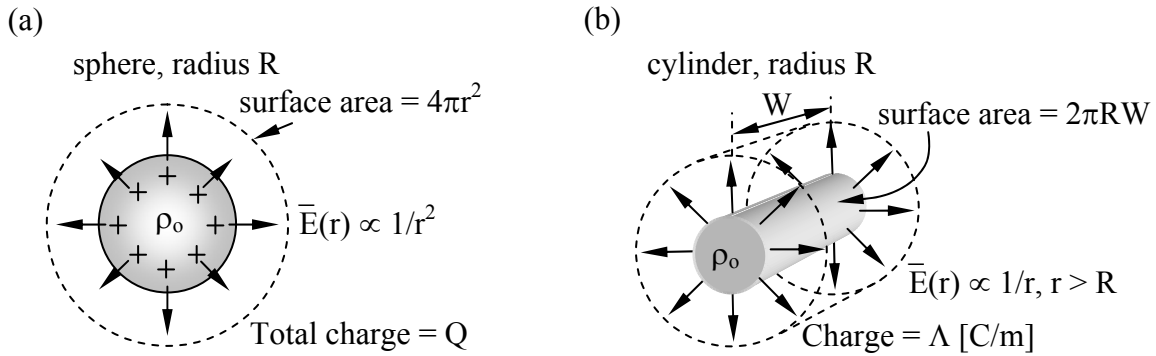


Figure 1.3.1 Electric fields $\bar{E}(r)$ produced by uniformly charged spheres and cylinders.

The symmetry of the solution must match the spherical symmetry of the problem, so \bar{E} must be independent of θ and ϕ , although it can depend on radius r . This symmetry requires that \bar{E} be radial and, more particularly:

$$\bar{E}(r, \theta, \phi) = \hat{r}E(r) \quad [\text{V/m}] \quad (1.3.3)$$

We can find $\bar{E}(r)$ by substituting (1.3.3) into (1.3.1). First consider $r > R$, for which (1.3.1) becomes:

$$4\pi r^2 \epsilon_0 E(r) = (4/3)\pi R^3 \rho_0 = Q \quad (1.3.4)$$

$$\bar{E}(r) = \hat{r} \frac{Q}{4\pi \epsilon_0 r^2} \quad (r > R) \quad (1.3.5)$$

Inside the sphere the same substitution into (1.3.1) yields:

$$4\pi r^2 \epsilon_0 E(r) = (4/3)\pi r^3 \rho_0 \quad (1.3.6)$$

$$\bar{E}(r) = \hat{r} \rho_o r / 3\epsilon_o \text{ [V/m]} \quad (r < R) \quad (1.3.7)$$

It is interesting to compare this dependence of \bar{E} on r with that for cylindrical geometries, which are also illustrated in Figure 1.3.1. We assume a uniform charge density of ρ_o within radius R , corresponding to Λ coulombs/meter. Substitution of (1.3.4) into (1.3.1) yields:

$$2\pi r W \epsilon_o E(r) = \pi R^2 \rho_o W = \Lambda W \text{ [C]} \quad (r > R) \quad (1.3.8)$$

$$\bar{E}(r) = \hat{r} \frac{\Lambda}{2\pi\epsilon_o r} = \hat{r} \frac{R^2 \rho_o}{2\epsilon_o r} \text{ [V/m]} \quad (r > R) \quad (1.3.9)$$

Inside the cylinder ($r < R$) the right-hand-side of (1.3.9) still applies, but with R^2 replaced with r^2 , so $\bar{E}(r) = \hat{r} r \rho_o / 2\epsilon_o$ instead.

To find the voltage difference, often called the difference in *electrical potential* Φ or the *potential difference*, between two points in space [V], we can simply integrate the static electric field $\bar{E} \cdot \hat{r}$ [V/m] along the field line \bar{E} connecting them. Thus in the spherical case the voltage difference $\Phi(r_1) - \Phi(r_2)$ between points at r_1 and at $r_2 > r_1$ is:

$$\Phi(r_1) - \Phi(r_2) = \int_{r_1}^{r_2} \bar{E} \cdot d\bar{r} = \frac{Q}{4\pi\epsilon_o} \int_{r_1}^{r_2} \frac{1}{r^2} \hat{r} \cdot d\bar{r} = -\frac{Q}{4\pi\epsilon_o r} \Big|_{r_1}^{r_2} = \frac{Q}{4\pi\epsilon_o} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \text{ [V]} \quad (1.3.10)$$

If we want to assign an absolute value to electrical potential or voltage V at a given location, we usually define the potential Φ to be zero at $r_2 = \infty$, so a spherical charge Q produces an electric potential $\Phi(r)$ for $r > R$ which is:

$$\Phi(r) = Q / 4\pi\epsilon_o r \text{ [V]} \quad (1.3.11)$$

The same computation for the cylindrical charge of Figure 1.3.1 and the field of (1.3.9) yields:

$$\Phi(r_1) - \Phi(r_2) = \int_{r_1}^{r_2} \bar{E} \cdot d\bar{r} = \frac{\Lambda}{2\pi\epsilon_o} \int_{r_1}^{r_2} \frac{1}{r} \hat{r} \cdot d\bar{r} = \frac{\Lambda \ln r}{2\pi\epsilon_o} \Big|_{r_1}^{r_2} = \frac{\Lambda}{2\pi\epsilon_o} \ln(r_2/r_1) \quad (1.3.12)$$

A third simple geometry is that of charged infinite parallel conducting plates separated by distance d , where the inner-facing surfaces of the upper and lower plates have surface charge density $+\rho_s$ and $-\rho_s$ [C/m²], respectively, as illustrated in Figure 1.3.2 for finite plates. The uniformity of infinite plates with respect to x , y , and ϕ requires that the solution \bar{E} also be independent of x , y , and ϕ . The symmetry with respect to ϕ requires that \bar{E} point in the $\pm z$ direction. Gauss's law (1.3.1) then requires that \bar{E} be independent of z because the integrals of \bar{D} over the top and bottom surfaces of any rectangular volume located between the plates must cancel since there is no charge within such a volume and no \bar{D} passing through its sides.

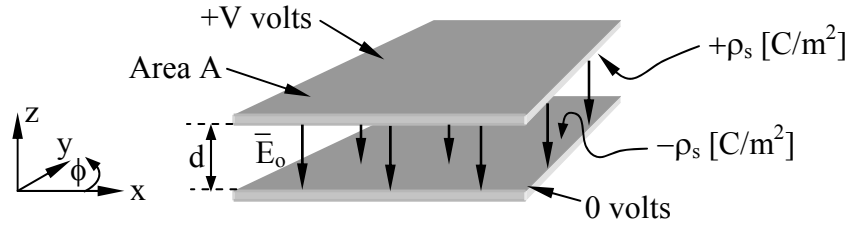


Figure 1.3.2 Electric field between parallel plates.

This solution for \bar{E} is consistent with the *rubber-band model* for field lines, which suggests that the excess positive and negative charges will be mutually attracted, and therefore will be pulled to the inner surfaces of the two plates, particularly if the gap d between the plates is small compared to their width. Gauss's Law (1.3.1) also tells us that the displacement vector \bar{D} integrated over a surface enclosing the entire structure must be zero because the integrated charge within that surface is zero; that is, the integrated positive charge, $\rho_s A$, balances the integrated negative charge, $-\rho_s A$ and \bar{D} external to the device can be zero everywhere. The electric potential difference V between the two plates can be found by integrating \bar{E} between the two plates. That is, $V = E_0 d$ volts for any path of integration, where $E_0 = \rho_s / \epsilon_0$ by Gauss's law.

Although the voltage difference between equipotentials can be computed by integrating along the electric field lines themselves, as done above, it is easy to show that the result does not depend on the path of integration. Assume there are two different paths of integration P_1 and P_2 between any two points of interest, and that the two resulting voltage differences are V_1 and V_2 . Now consider the closed contour C of integration that is along path P_1 in the positive direction and along P_2 in the reverse direction so as to make a closed loop. Since this contour integral must yield zero, as shown below in (1.3.13) using Faraday's law for the static case where $\partial/\partial t = 0$, it follows that $V_1 = V_2$ and that all paths of integration yield the same voltage difference.

$$V_1 - V_2 = \int_{P_1} \bar{E} \cdot d\bar{s} - \int_{P_2} \bar{E} \cdot d\bar{s} = \oint_C \bar{E} \cdot d\bar{s} = -\frac{\partial}{\partial t} \iint_A \bar{B} \cdot d\bar{a} = 0 \quad (1.3.13)$$

In summary, electric fields decay as $1/r^2$ from spherical charge concentrations, as $1/r$ from cylindrical ones, and are uniform in planar geometries. The corresponding electric potentials decay as $1/r$, $-\ln r$, and x , respectively, as a result of integration over distance. The potential Φ for the cylindrical case becomes infinite as $r \rightarrow \infty$ because the cylinder is infinitely long; the expression for the potential difference between concentric cylinders of finite radius is valid, however. Within both uniform spherical and cylindrical charge distributions the electric field increases from zero linearly with radius r . In each case the electric field distribution is explained by the rubber-band model in which the rubber bands (field lines) repel each other laterally while being pulled on by opposite electric charges.

It is extremely useful to note that Maxwell's equations are linear, so that superposition applies. That is, the total electric field \bar{E} equals that due to the sum of all charges present, where the contribution to \bar{E} from each charge Q is given by (1.3.5). Electric potentials Φ also superimpose, where the contribution from each charge Q is given by (1.3.11).

1.4 Ampere's Law and magnetostatic fields

The relevant Maxwell's equation for static current densities $\bar{\mathbf{J}}$ [A/m²] is Ampere's law, which says that for time-invariant cases the integral of magnetic field $\bar{\mathbf{H}}$ around any closed contour in a right-hand sense equals the area integral of current density $\bar{\mathbf{J}}$ [A/m²] flowing through that contour:

$$\oint_c \bar{\mathbf{H}} \cdot d\bar{\mathbf{s}} = \iint_A \bar{\mathbf{J}} \cdot d\bar{\mathbf{a}} \quad (1.4.1)$$

Figure 1.4.1 illustrates a simple cylindrical geometry for which we can readily compute $\bar{\mathbf{H}}$ produced by current I ; the radius of the cylinder is R and the uniform current density flowing through it is J_0 [A/m²]. The cylinder is infinitely long.

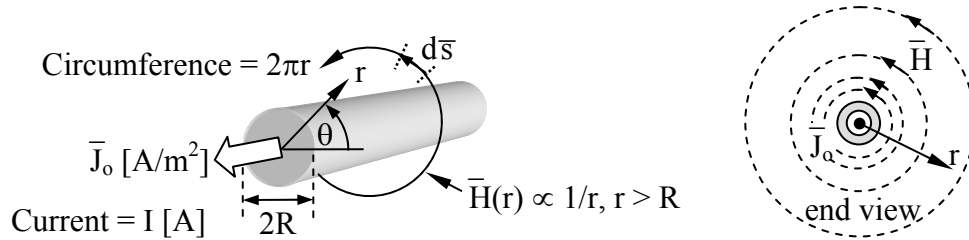


Figure 1.4.1 Magnetic field produced by a uniform cylindrical current.

Because the problem is cylindrically symmetric (not a function of θ), and uniform with respect to the cylindrical axis z , so is the solution. Thus $\bar{\mathbf{H}}$ depends only upon radius r . Substitution of $\bar{\mathbf{H}}(r)$ into (1.4.1) yields:

$$\int_0^{2\pi} \bar{\mathbf{H}}(r) \cdot \hat{\theta} r d\theta = \int_0^{2\pi} \int_0^R J_0 r dr d\theta = J_0 \pi R^2 = I \text{ [A]} \quad (1.4.2)$$

where the total current I is simply the uniform current density J_0 times the area πR^2 of the cylinder. The left-hand-side of (1.4.2) simply equals $H(r)$ times the circumference of a circle of radius r , so (1.4.2) becomes:

$$\bar{\mathbf{H}}(r) = \hat{\theta} \frac{I}{2\pi r} = \hat{\theta} \frac{J_0 \pi R^2}{2\pi r} \text{ [A/m]} \quad (r > R) \quad (1.4.3)$$

Within the cylindrical wire where $r < R$, (1.4.2) becomes:

$$H(r) 2\pi r = \int_0^{2\pi} \int_0^r J_0 r dr d\theta = J_0 \pi r^2 \quad (1.4.4)$$

$$\bar{\mathbf{H}}(r) = \hat{\theta} J_0 r / 2 \text{ [A/m]} \quad (r < R) \quad (1.4.5)$$

Therefore $H(r)$ increases linearly with r within the wire and current distribution, and is continuous at $r = R$, where both (1.4.3) and (1.4.5) agree that $H(r) = J_0 R/2$.

Another simple geometry involves parallel plates. Assume equal and opposite current densities, J_s [A/m], flow in infinite parallel plates separated by distance d , as illustrated in Figure 1.4.2 for finite plates. The integral of Ampere's law (1.4.1) around any contour C_1 circling both plates is zero because the net current through that contour is zero. A non-zero integral would require an external source of field, which we assume does not exist here. Thus \bar{H} above and below the plates is zero. Since the integral of (1.4.1) around any contour C_2 that circles the upper plate yields $H_x W = J_s W$, where the x component of the magnetic field anywhere between the plates is $H_x = J_s$ [A/m]; thus the magnetic field \bar{H} between the plates is uniform. An integral around any contour in any y - z plane would circle no net current, so $H_z = 0$, and a similar argument applies to H_y , which is also zero. This configuration is discussed further in Section 3.2.1.

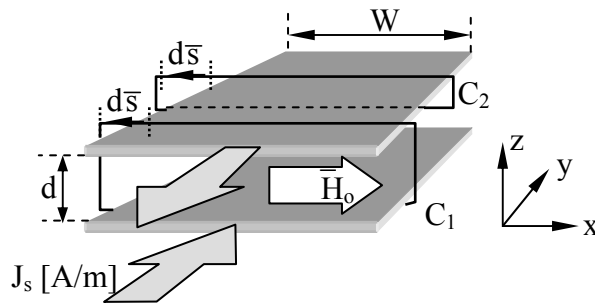


Figure 1.4.2 Static magnetic field between parallel plates.

More generally, because Maxwell's equations are linear, the total magnetic field \bar{H} at any location is the integral of contributions made by current densities \bar{J} nearby. Section 10.1 proves the Biot-Savart law (1.4.6), which defines how a current distribution \bar{J}' at position \bar{r}' within volume V' contributes to \bar{H} at position \bar{r} :

$$\bar{H}(\bar{r}, t) = \iiint_{V'} \frac{\bar{J}' \times (\bar{r} - \bar{r}')}{4\pi |\bar{r} - \bar{r}'|^3} dv' \quad (\text{Biot-Savart law}) \quad (1.4.6)$$

To summarize, electric and magnetic fields are simple fictions that explain all electromagnetic behavior as characterized by Maxwell's equations and the Lorentz force law, which are examined further in Chapter 2. A simple physical model for the static behavior of electric fields is that of rubber bands that tend to pull opposite electric charges toward one another, but that tend to repel neighboring field lines laterally. Static magnetic fields behave similarly, except that the role of magnetic charges (which have not been shown to exist) is replaced by current loops acting as magnetic dipoles in ways that are discussed later.

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