Chapter 5: Electromagnetic Forces

5.1 *Forces on free charges and currents*

5.1.1 Lorentz force equation and introduction to force

The Lorentz force equation (1.2.1) fully characterizes electromagnetic forces on stationary and moving charges. Despite the simplicity of this equation, it is highly accurate and essential to the understanding of all electrical phenomena because these phenomena are observable only as a result of forces on charges. Sometimes these forces drive motors or other actuators, and sometimes they drive electrons through materials that are heated, illuminated, or undergoing other physical or chemical changes. These forces also drive the currents essential to all electronic circuits and devices.

When the electromagnetic fields and the location and motion of free charges are known, the calculation of the forces acting on those charges is straightforward and is explained in Sections 5.1.2 and 5.1.3. When these charges and currents are confined within conductors instead of being isolated in vacuum, the approaches introduced in Section 5.2 can usually be used. Finally, when the charges and charge motion of interest are bound within stationary atoms or spinning charged particles, the Kelvin force density expressions developed in Section 5.3 must be added. The problem usually lies beyond the scope of this text when the force-producing electromagnetic fields are not given but are determined by those same charges on which the forces are acting (e.g., plasma physics), and when the velocities are relativistic.

The simplest case involves the forces arising from known electromagnetic fields acting on free charges in vacuum. This case can be treated using the *Lorentz force equation* (5.1.1) for the *force vector* $\overline{f}$ acting on a charge q [Coulombs]:

$$\overline{f} = q\left(\overline{E} + \overline{v} \times \mu_o \overline{H}\right) \quad \text{[Newtons]} \qquad \text{(Lorentz force equation)} \qquad (5.1.1)$$

where $\overline{E}$ and $\overline{H}$ are the local electric and magnetic fields and $\overline{v}$ is the charge velocity vector [m s^{-1}].

5.1.2 Electric Lorentz forces on free electrons

The *cathode-ray tube* (CRT) used for displays in older computers and television sets, as illustrated in Figure 5.1.1, provides a simple example of the Lorentz force law (5.1.1). Electrons thermally excited by a heated *cathode* at -V volts escape at low energy and are accelerated in vacuum at acceleration $\overline{a}$ [m s^{-2}] toward the grounded *anode* by the electric field $\overline{E} \cong -\hat{z}V/s$ between anode and cathode[13]; V and s are the voltage across the tube and the cathode-anode

[13] The anode is grounded for safety reasons; it lies at the tube face where users may place their fingers on the other side of the glass faceplate. Also, the cathode and anode are sometimes shaped so that the electric field $\overline{E}$, the force $\overline{f}$, and the acceleration $\overline{a}$ are functions of z instead of being constant; i.e., $\overline{E} \neq -\hat{z}V/D$.

separation, respectively. In electronics the anode always has a more positive potential Φ than the cathode, by definition.

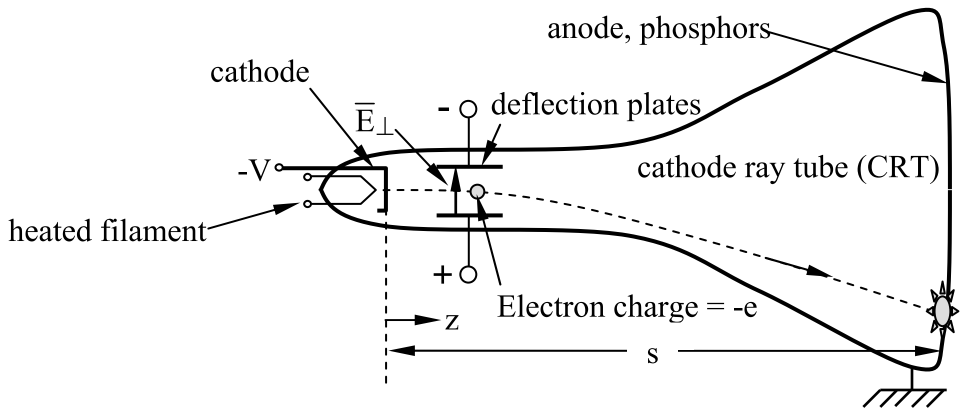

Figure 5.1.1 Cathode ray tube.

The acceleration $\overline{a}$ is governed by *Newton's law*:

$$\overline{f} = m\overline{a} \qquad \text{(Newton's law)} \qquad (5.1.2)$$

where m is the mass of the unconstrained accelerating particle. Therefore the acceleration a of the electron charge q = -e in an electric field E = V/s is:

$$a = f/m = qE/m \cong eV/ms \quad [ms^{-2}] \qquad (5.1.3)$$

The subsequent velocity $\overline{v}$ and position z of the particle can be found by integration of the acceleration $\hat{z}a$:

$$\overline{v} = \int_0^t \overline{a}(t)dt = \overline{v}_o + \hat{z}at \quad [ms^{-1}] \qquad (5.1.4)$$

$$z = z_o + \hat{z} \bullet \int_0^t \overline{v}(t)dt = z_o + \hat{z} \bullet \overline{v}_o t + at^2/2 \quad [m] \qquad (5.1.5)$$

where we have defined the initial electron position and velocity at t = 0 as z_o and $\overline{v}_o$, respectively.

The increase w_k in the kinetic energy of the electron equals the accumulated work done on it by the electric field $\overline{E}$. That is, the increase in the kinetic energy of the electron is the product of the constant force f acting on it and the distance s the electron moved in the direction of $\overline{f}$ while experiencing that force. If s is the separation between anode and cathode, then:

$$w_k = fs = (eV/s)s = eV \quad [J] \qquad (5.1.6)$$

Thus the kinetic energy acquired by the electron in moving through the potential difference V is eV Joules. If V = 1 volt, then w_k is one "*electron volt*", or "e" Joules, where $e \cong 1.6 \times 10^{-19}$ Coulombs. The kinetic energy increase equals eV even when $\overline{E}$ is a function of z because:

$$w_k = \int_0^D eE_z dz = eV \tag{5.1.7}$$

Typical values for V in television CRT's are generally less than 50 kV so as to minimize dangerous x-rays produced when the electrons impact the phosphors on the CRT faceplate, which is often made of x-ray-absorbing leaded glass.

Figure 5.1.1 also illustrates how time-varying lateral electric fields $\overline{E}_{\perp}(t)$ can be applied by deflection plates so as to scan the electron beam across the CRT faceplate and "paint" the image to be displayed. At higher tube voltages V the electrons move so quickly that the lateral electric forces have no time to act, and magnetic deflection is used instead because lateral magnetic forces increase with electron velocity v.

Example 5.1A

Long interplanetary or interstellar voyages might eject charged particles at high speeds to obtain thrust. What particles are most efficient at imparting total momentum if the rocket has only E joules and M kg available to expend for this purpose?

Solution: Particles of charge e accelerating through an electric potential of V volts acquire energy eV [J] = $mv^2/2$; such energies can exceed those available in chemical reactions. The total increase in rocket momentum = nmv [N], where n is the total number of particles ejected, m is the mass of each particle, and v is their velocity. The total mass and energy available on the rocket is M = nm and E = neV. Since v = $(2eV/m)^{0.5}$, the total momentum ejected is Mv = nmv = $(n^2 2eVm)^{0.5} = (2EM)^{0.5}$. Thus any kind of charged particles can be ejected, only the total energy E and mass ejected M matter.

5.1.3 Magnetic Lorentz forces on free charges

An alternate method for laterally scanning the electron beam in a CRT utilizes magnetic deflection applied by coils that produce a magnetic field perpendicular to the electron beam, as illustrated in Figure 5.1.2. The magnetic Lorentz force on the charge q = -e (1.6021×10^{-19} Coulombs) is easily found from (5.1.1) to be:

$$\overline{f} = -e\overline{v} \times \mu_o \overline{H} \quad [N] \tag{5.1.8}$$

Thus the illustrated CRT electron beam would be deflected upwards, where the magnetic field $\overline{H}$ produced by the coil is directed out of the paper; the magnitude of the force on each electron is $ev\mu_o H$ [N].

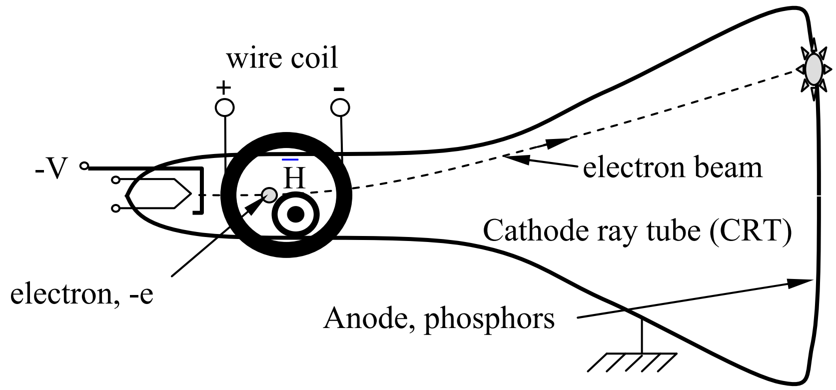

Figure 5.1.2 Magnetic deflection of electrons in a cathode ray tube.

The lateral force on the electrons $ev\mu_oH$ can be related to the CRT voltage V. Electrons accelerated from rest through a potential difference of V volts have kinetic energy eV [J], where:

$$eV = mv^2/2 \tag{5.1.9}$$

Therefore the electron velocity $v = (2eV/m)^{0.5}$, where m is the electron mass (9.107×10^{-31} kg), and the lateral deflection increases with tube voltage V, whereas it decreases if electrostatic deflection is used instead.

Another case of magnetic deflection is illustrated in Figure 5.1.3 where a free electron moving perpendicular to a magnetic field $\overline{B}$ experiences a force $\overline{f}$ orthogonal to its velocity vector $\overline{v}$, since $\overline{f} = q\overline{v}\times\mu_o\overline{H}$.

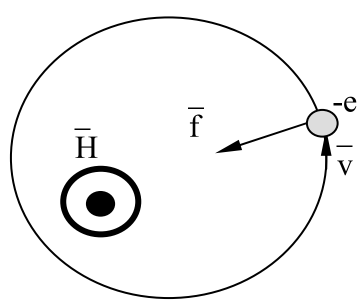

Figure 5.1.3 Cyclotron motion of an electron.

This force $|\overline{f}|$ is always orthogonal to $\overline{v}$ and therefore the trajectory of the electron will be circular with radius R at angular frequency ω_e [radians s^{-1}]:

$$|f| = ev\mu_oH = m_ea = m_e\omega_e{}^2R = m_ev\omega_e \tag{5.1.10}$$

where $v = \omega_eR$. We can solve (5.1.9) for this "electron *cyclotron frequency*" ω_e:

$$\omega_e = e\mu_o H/m_e \qquad \text{(electron cyclotron frequency)} \qquad (5.1.11)$$

which is independent of v and the electron energy, provided the electron is not relativistic. Thus the magnitudes of magnetic fields can be measured by observing the radiation frequency ω_e of free electrons in the region of interest.

Example 5.1B

What is the radius r_e of cyclotron motion for a 100 e.v. free electron in the terrestrial magnetosphere[14] where $B \cong 10^{-6}$ Tesla? What is the radius r_p for a free proton with the same energy? The masses of electrons and protons are $\sim 9.1\times10^{-31}$ and 1.7×10^{-27} kg, respectively.

Solution: The magnetic force on a charged particle is $qv\mu_o H = ma = mv^2/r$, where the velocity v follows from (5.1.9): $eV = mv^2/2 \Rightarrow v = (2eV/m)^{0.5}$. Solving for r_e yields $r_e = m_e v/e\mu_o H = (2Vm/e)^{0.5}/\mu_o H \cong (2\times100\times9.1\times10^{-31}/1.6\times10^{-19})^{0.5}/10^{-6} \cong 34$ m for electrons and ~2.5 km for protons.

5.2 Forces on charges and currents within conductors

5.2.1 Electric Lorentz forces on charges within conductors

Static electric forces on charges within conductors can also be calculated using the Lorentz force equation (5.1.1), which becomes $\overline{f} = q\overline{E}$. For example, consider the capacitor plates illustrated in Figure 5.2.1(a), which have total surface charges of ±Q coulombs on the two conductor surfaces facing each other. The fields and charges for capacitor plates were discussed in Section 3.1.3.

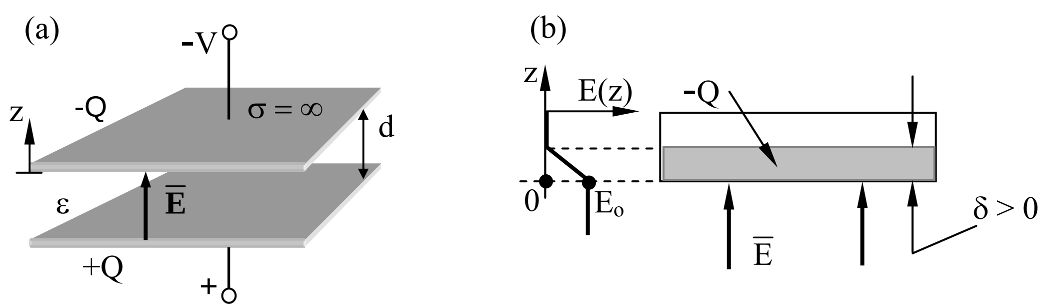

Figure 5.2.1 Charge distribution within conducting capacitor plates.

To compute the total attractive *electric pressure* P_e [N m^{-2}] on the top plate, for example, we can integrate the Lorentz force density $\overline{F}$ [N m^{-3}] acting on the charge distribution $\rho(z)$ over depth z and unit area:

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[14] The magnetosphere extends from the ionosphere to several planetary radii; particle collisions are rare compared to the cyclotron frequency.

$$\overline{F} = \rho\overline{E} \quad [N\ m^{-3}] \tag{5.2.1}$$

$$\overline{P}_e = \int_0^{\infty}\overline{F}(z)dz = \hat{z}\int_0^{\infty}\rho(z)E_z(z)dz \quad [N\ m^{-2}] \tag{5.2.2}$$

where we have defined $\overline{E} = \hat{z}E_z$, as illustrated.

Care is warranted, however, because surface the charge $\rho(z)$ is distributed over some infinitesimal depth δ, as illustrated in Figure 5.2.1(b), and those charges at greater depths are shielded by the others and therefore see a smaller electric field $\overline{E}$. If we assume $\varepsilon = \varepsilon_o$ inside the conductors and a planar geometry with $\partial/\partial x = \partial/\partial y = 0$, then Gauss's law, $\nabla \bullet \varepsilon\overline{E} = \rho$, becomes:

$$\varepsilon_o dE_z/dz = \rho(z) \tag{5.2.3}$$

This expression for $\rho(z)$ can be substituted into (5.2.2) to yield the pressure exerted by the electric fields on the capacitor plate and perpendicular to it:

$$P_e = \int_{E_o}^{0}\varepsilon_o dE_z E_z = -\varepsilon_o E_o^{\,2}/2 \qquad \text{(electric pressure on conductors)} \tag{5.2.4}$$

The charge density ρ and electric field E_z are zero at levels below δ, and the field strength at the surface is E_o. If the conductor were a dielectric with $\varepsilon \neq \varepsilon_o$, then the Kelvin polarization forces discussed in Section 5.3.2 would also have to be considered.

Thus the electric pressure P_e [N m^{-2}] pulling on a charged conductor is the same as the immediately adjacent electric energy density [Jm^{-3}], and is independent of the sign of ρ and $\overline{E}$. These dimensions are identical because [J] = [Nm]. The maximum achievable electric field strength thus limits the maximum achievable electric pressure P_e, which is negative because it pulls rather than pushes conductors.

An alternate form for the electric pressure expression is:

$$P_e = -\varepsilon_o E_o^{\,2}/2 = -\rho_s^{\,2}/2\varepsilon_o \quad [Nm^{-2}] \quad \text{(electric pressure on conductors)} \tag{5.2.5}$$

where ρ_s is the surface charge density [cm^{-2}] on the conductor and ε_o is its permittivity; boundary conditions at the conductor require $D = \varepsilon_o E = \sigma_s$. Therefore if the conductor were adjacent to a dielectric slab with $\varepsilon \neq \varepsilon_o$, the electrical pressure on the conductor would still be determined by the surface charge, electric field, and permittivity ε_o within the conductor; the pressure does not otherwise depend on ε of adjacent rigid materials.

We can infer from (5.2.4) the intuitively useful result that the average electric field pulling on the charge Q is E/2 since the total pulling force f = - P_eA, where A is the area of the plate:

$$f = -P_e A = A\varepsilon_o E^2/2 = AD(E/2) = Q(E/2) \tag{5.2.6}$$

If the two plates were both charged the same instead of oppositely, the surface charges would repel each other and move to the outer surfaces of the two plates, away from each other. Since there would now be no E between the plates, it could apply no force. However, the charges Q on the outside are associated with the same electric field strength as before, $E = Q/\varepsilon_o A$. These electric fields outside the plates therefore pull them apart with the same force density as before, $P_e = -\varepsilon_o E^2/2$, and the force between the two plates is now repulsive instead of attractive. In both the attractive and repulsive cases we have assumed the plate width and length are sufficiently large compared to the plate separation that fringing fields can be neglected.

Example 5.2A

Some copy machines leave the paper electrically charged. What is the electric field E between two adjacent sheets of paper if they cling together electrically with a force density of 0.01 oz. ≅ 0.0025 N per square centimeter = 25 Nm^{-2}? If we slightly separate two such sheets of paper by 4 cm, what is the voltage V between them?

Solution: Electric pressure is $P_e = -\varepsilon_o E^2/2$ [N m^{-2}], so $E = (-2P_e/\varepsilon_o)^{0.5} = (2\times 25/8.8\times 10^{-12})^{0.5} = 2.4$ [MV/m]. At 4 cm distance this field yields ~95 kV potential difference between the sheets. The tiny charge involved renders this voltage harmless.

5.2.2 Magnetic Lorentz forces on currents in conductors

The Lorentz force law can also be used to compute forces on electrons moving within conductors for which $\mu = \mu_o$. Computation of forces for the case $\mu \neq \mu_o$ is treated in Sections 5.3.3 and 5.4. If there is no net charge and no current flowing in a wire, the forces on the positive and negative charges all cancel because the charges comprising matter are bound together by strong inter- and intra-atomic forces.

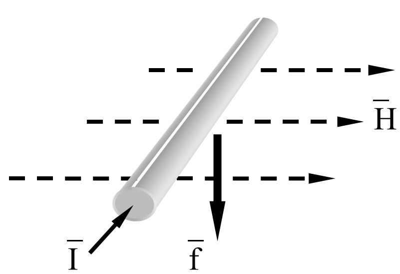

Figure 5.2.2 Magnetic force on a current-carrying wire.

However, if n_1 carriers per meter of charge q are flowing in a wire[15], as illustrated in Figure 5.2.2, then the total force density $\overline{F} = n_1 q\overline{v} \times \mu_o \overline{H} = \overline{I} \times \mu_o \overline{H}$ $[N\ m^{-1}]$ exerted by a static magnetic field $\overline{H}$ acting on the static current $\overline{I}$ flowing in the wire is:

$$\overline{F} = n_1 q\overline{v} \times \mu_o \overline{H} = \overline{I} \times \mu_o \overline{H} \quad [N\ m^{-1}] \qquad \text{(magnetic force density on a wire)} \qquad (5.2.7)$$

where $\overline{I} = n_1 q\overline{v}$. If $\overline{H}$ is uniform, this force is not a function of the cross-section of the wire, which could be a flat plate, for example.

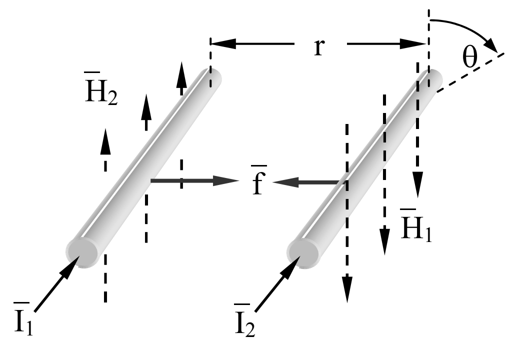

Figure 5.2.3 Magnetic forces attracting parallel currents.

We can easily extend the result of (5.2.7) to the case of two parallel wires carrying the same current I in the same $+\hat{z}$ direction and separated by distance r, as illustrated in Figure 5.2.3. Ampere's law with cylindrical symmetry readily yields $\overline{H}(r) = \hat{\theta}H(r)$:

$$\oint_c \overline{H} \bullet d\overline{s} = I = 2\pi r H \Rightarrow H = I/2\pi r \qquad (5.2.8)$$

The force density $\overline{F}$ pulling the two parallel wires together is then found from (5.2.7) and (5.2.8) to be:

$$|\overline{F}| = \mu_o I^2 / 2\pi r \quad [Nm^{-1}] \qquad (5.2.9)$$

The simplicity of this equation and the ease of measurement of F, I, and r led to its use in defining the permeability of free space, $\mu_o = 4\pi \times 10^{-7}$ Henries/meter, and hence the definition of a *Henry* (the unit of inductance). If the two currents are in opposite directions, the force acting on the wires is repulsive. For example, if I = 10 amperes and r = 2 millimeters, then (5.2.9) yields F = $4\pi \times 10^{-7} \times 10^2/2\pi 2 \times 10^{-3}$ = 0.01 Newtons/meter; this is approximately the average repulsive force between the two wires in a 120-volt AC lamp cord delivering one kilowatt. These forces are attractive when the currents are parallel, so if we consider a single wire as consisting of

[15] The notation n_j signifies number density $[m^{-j}]$, so n_1 and n_3 indicate numbers per meter and per cubic meter, respectively.

parallel strands, they will squeeze together due to this *pinch effect*. At extreme currents, these forces can actually crush wires, so the maximum achievable instantaneous current density in wires is partly limited by their mechanical strength. The same effect can pinch electron beams flowing in charge-neutral plasmas.

The magnetic fields associated with surface currents on flat conductors generally exert a pressure $\overline{P}$ [N m^{-2}] that is simply related to the instantaneous field strength $|\overline{H}_s|$ at the conductor surface. First we can use the magnetic term in the Lorentz force law (5.2.7) to compute the force density $\overline{F}$ [N m^{-3}] on the surface current $\overline{J}_s$ [A m^{-1}]:

$$\overline{F} = nq\overline{v} \times \mu_o \overline{H} = \overline{J} \times \mu_o \overline{H} \quad \left[\text{N m}^{-3}\right] \tag{5.2.10}$$

where n is the number of charges q per cubic meter. To find the *magnetic pressure* P_m [N m^{-2}] on the conductor we must integrate the force density $\overline{F}$ over depth z, where both $\overline{J}$ and $\overline{H}$ are functions of z, as governed by Ampere's law in the static limit:

$$\nabla \times \overline{H} = \overline{J} \tag{5.2.11}$$

If we assume $\overline{H} = \hat{y}H_y(z)$ then $\overline{J}$ is in the x direction and $\partial H_y/\partial x = 0$, so that:

$$\begin{aligned} \nabla \times \overline{H} &= \hat{x}\left(\partial H_z/\partial y - \partial H_y/\partial z\right) + \hat{y}\left(\partial H_x/\partial z - \partial H_z/\partial x\right) + \hat{z}\left(\partial H_y/\partial x - \partial H_x/\partial y\right) \\ &= -\hat{x}dH_y/dz = \hat{x}J_x(z) \end{aligned} \tag{5.2.12}$$

The instantaneous magnetic pressure P_m exerted by H can now be found by integrating the force density equation (5.2.10) over depth z to yield:

$$\begin{aligned} \overline{P}_m &= \int_0^\infty \overline{F}\, dz = \int_0^\infty \overline{J}(z) \times \mu_o \overline{H}(z)\, dz = \int_0^\infty \left[-\hat{x}dH_y/dz\right] \times \left[\hat{y}\mu_o H_y(z)\right] dz \\ \overline{P}_m &= -\hat{z}\mu_o \int_H^0 H_y\, dH_y = \hat{z}\mu_o H^2/2 \quad \left[\text{Nm}^{-2}\right] \qquad \text{(magnetic pressure)} \end{aligned} \tag{5.2.13}$$

We have assumed $\overline{H}$ decays to zero somewhere inside the conductor. As in the case of the electrostatic pull of an electric field on a charged conductor, the average field strength experienced by the surface charges or currents is half that at the surface because the fields inside the conductor are partially shielded by any overlying charges or currents. The time average magnetic pressure for sinusoidal H is $\langle P_m \rangle = \mu_o |\underline{\overline{H}}|^2/4$.

5.3 Forces on bound charges within materials

5.3.1 Introduction

Forces on materials can be calculated in three different ways: 1) via the Lorentz force law, as illustrated in Section 5.2 for free charges within materials, 2) via energy methods, as illustrated in Section 5.4, and 3) via photonic forces, as discussed in Section 5.6. When polarized or magnetized materials are present, as discussed here in Section 5.3, the Lorentz force law must be applied not only to the free charges within the materials, i.e. the surface charges and currents discussed earlier, but also to the orbiting and spinning charges bound within atoms. When the Lorenz force equation is applied to these bound charges, the result is the Kelvin polarization and magnetization force densities. Under the paradigm developed in this chapter these Kelvin forces must be added to the Lorentz forces on the free charges[16]. The Kelvin force densities are non-zero only when inhomogeneous fields are present, as discussed below in Sections 5.3.2 and 5.3.3. But before discussing Kelvin forces it is useful to review the relationship between the Lorentz force law and matter.

The Lorentz force law is complete and exact if we ignore relativistic issues associated with either extremely high velocities or field strengths; neither circumstance is relevant to current commercial products. To compute all the Lorentz forces on matter we must recognize that classical matter is composed of atoms comprised of positive and negative charges, some of which are moving and exhibit magnetic moments due to their spin or orbital motions. Because these charges are trapped in the matter, any forces on them are transferred to that matter, as assumed in Section 5.2 for electric forces on surface charges and for magnetic forces on surface currents.

When applying the Lorentz force law within matter under our paradigm it is important to use the expression:

$$\overline{f} = q\left(\overline{E} + \overline{v} \times \mu_o \overline{H}\right) \quad [\text{Newtons}] \tag{5.3.1}$$

without substituting $\mu\overline{H}$ for the last term when $\mu \neq \mu_o$. A simple example illustrates the dangers of this common notational shortcut. Consider the instantaneous magnetic pressure (5.2.13) derived using the Lorentz force law for a uniform plane wave normally incident on a conducting plate having $\mu \neq \mu_o$. The same force is also found later in (5.6.5) using photon momentum. If we incorrectly use:

$$\overline{f} = q\left(\overline{E} + \overline{v} \times \mu\overline{H}\right) \quad [\text{Newtons}] \qquad \text{(incorrect for this example)} \tag{5.3.2}$$

[16] The division here between Lorentz forces acting on free charges and the Lorentz forces acting on bound charges (often called Kelvin forces) is complete and accurate, but not unique, for these forces can be grouped and labeled differently, leading to slightly different expressions that are also correct.

because $\overline{v}$ occurs within μ, then the computed wave pressure would increase with μ, whereas the photon model has no such dependence and yields $P_m = \mu_o H^2/2$, the same answer as does (5.2.13). The photon model depends purely on the input and output photon momentum fluxes observed some distance from the mirror, and thus the details of the mirror construction are irrelevant once the fraction of photons reflected is known.

This independence of the Lorentz force from μ can also be seen directly from the Lorentz force calculation that led to (5.2.13). In this case the total surface current is not a function of μ for a perfect reflector, and neither is $\overline{H}$ just below the surface; they depend only upon the incident wave and the fact that the mirror is nearly perfect. $\overline{H}$ does decay faster with depth when μ is large, as discussed in Section 9.3, but the average $|\overline{H}|$ experienced by the surface-current electrons is still half the value of $\overline{H}$ at the surface, so $\overline{f}$ is unchanged as μ varies. The form of the Lorentz force law presented in (5.3.2) can therefore be safely used under our force paradigm only when $\mu = \mu_o$, although the magnetic term is often written as $\overline{v} \times \overline{B}$.

There are alternate correct paradigms that use μ in the Lorentz law rather than μ_o, but they interpret Maxwell's equations slightly differently. These alternative approaches are not discussed here.

The Lorentz force law can also be applied to those cases where non-uniform fields pull on dielectrics or permeable materials, as suggested by Figure 5.3.1. These problems are often more easily solved, however, using energy (Section 5.4) or pressure (Section 5.5) methods. To compute in general the forces on matter exerted by non-uniform electric or magnetic fields we can derive the Kelvin polarization and magnetization force density expressions from the Lorentz equation, as shown in Sections 5.3.2 and 5.3.3, respectively.

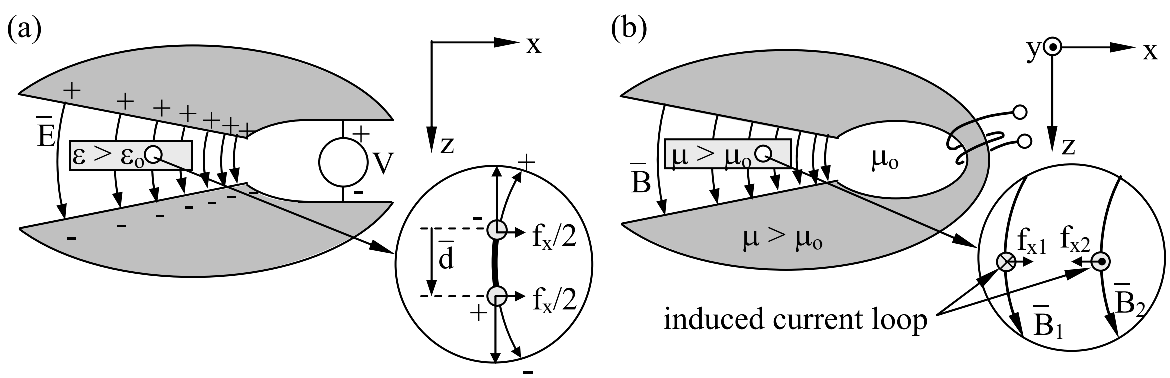

Figure 5.3.1 Kelvin polarization and magnetization forces on materials.

The derivations of the Kelvin force density expressions are based on the following simple models for charges in matter. Electric Lorentz forces act on atomic nuclei and the surrounding electron clouds that are bound together, and on any free charges. The effect of $\overline{E}$ on positive and negative charges bound within an atom is to displace their centers slightly, inducing a small electric dipole. The resulting atomic electric dipole moment is:

$$\overline{p} = \overline{d}q \qquad \text{(Coulomb meters)} \qquad (5.3.3)$$

where $\overline{d}$ is the displacement vector [m] pointing from the negative charge center to the positive charge center for each atom, and q is the atomic charge or atomic number. As discussed further in Section 5.3.2, Kelvin polarization forces result when the field gradients cause the electric field lines to curve slightly so that the directions of the electric Lorentz forces are slightly different for the two ends of the field-induced electric dipoles so they do not cancel exactly, leaving a net residual force.

The magnetic Lorentz forces act on electrons classically orbiting atomic nuclei with velocities $\overline{v}$, and act on electrons with classical charge densities spinning at velocity $\overline{v}$ about the electron spin axis. Protons also spin, and therefore both electrons and protons possess magnetic dipole moments; these spin moments are smaller than those due to electron orbital motion. If we consider these spin and orbital motions as being associated with current loops, then we can see that the net force on such a loop would be non-zero if the magnetic fields perpendicular to these currents were different on the two sides of the loop. Such differences exist when the magnetic field has a non-zero gradient and then Kelvin magnetization forces result, as discussed in Section 5.3.3. The electromagnetic properties of matter are discussed further in Sections 2.5 and 9.5.

5.3.2 Kelvin polarization force density

Kelvin polarization forces result when a non-zero electric field gradient causes the Lorentz electric forces on the two charge centers of each induced electric dipole in a dielectric to differ, as illustrated in Figure 5.3.1(a). The force density can be found by summing the force imbalance vectors for each dipole within a unit volume.

Assume the center of the negative charge -q for a particular atom is at $\overline{r}$, and the center of the positive charge +q is at $\overline{r}+\overline{d}$. Then the net electric Lorentz force on that atom in the x direction is:

$$f_x = q\left[E_x\left(\overline{r}+\overline{d}\right) - E_x\left(\overline{r}\right)\right] = q\left(\overline{d}\bullet\nabla E_x\right) \quad [\text{N}] \qquad (5.3.4)$$

Thus $\overline{f}_x$ is the projection of the charge offset $\overline{d}$ on the gradient of qE_x. We recall $\nabla \equiv \hat{x}\,\partial/\partial x + \hat{y}\,\partial/\partial y + \hat{z}\,\partial/\partial z$.

Equation (5.3.4) can easily be generalized to:

$$\overline{f} = \hat{x}\left(q\overline{d}\bullet\nabla E_x\right) + \hat{y}\left(q\overline{d}\bullet\nabla E_y\right) + \hat{z}\left(q\overline{d}\bullet\nabla E_z\right) \qquad (5.3.5)$$

$$= \hat{x}\left(\overline{p}\bullet\nabla E_x\right) + \hat{y}\left(\overline{p}\bullet\nabla E_y\right) + \hat{z}\left(\overline{p}\bullet\nabla E_z\right) \equiv \overline{p}\bullet\nabla\overline{E} \quad [\text{N}] \qquad (5.3.6)$$

where $\overline{p} = q\overline{d}$ and (5.3.6) defines the new compact notation $\overline{p} \bullet \nabla\overline{E}$. Previously we have defined only $\nabla \times \overline{E}$ and $\nabla \bullet \overline{E}$, and the notation $\overline{p} \bullet [.]$ would have implied a scalar, not a vector. Thus the new operator defined here is [$\bullet\nabla$], and it operates on a pair of vectors to produce a vector.

Equation (5.3.6) then yields the Kelvin polarization force density $\overline{F}_p = n\overline{f}$, where n is the density of atomic dipoles [m^{-3}], and the polarization density of the material $\overline{P}$ is $n\overline{p}$ [$C\ m^{-2}$]:

$$\overline{F}_p = \overline{P} \bullet \nabla\overline{E} \quad \left[N\ m^{-3}\right] \qquad \text{(Kelvin polarization force density)} \qquad (5.3.7)$$

Equation (5.3.7) states that electrically polarized materials are pulled into regions having stronger electric fields if there is polarization $\overline{P}$ in the direction of the gradient. Less obvious from (5.3.7) is the fact that there can be such a force even when the applied electric field $\overline{E}$ and $\overline{P}$ are orthogonal to the field gradient, as illustrated in Figure 5.3.1(a). In this example a z-polarized dielectric is drawn in the x direction into regions of stronger E_z. This happens in curl-free fields because then a non-zero $\partial E_z/\partial x$ implies a non-zero $\partial E_x/\partial z$ that contributes to $\overline{F}_p$. This relation between partial derivatives follows from the definition:

$$\nabla \times \overline{E} = 0 = \hat{x}\left(\partial E_z/\partial y - \partial E_y/\partial z\right) + \hat{y}\left(\partial E_x/\partial z - \partial E_z/\partial x\right) + \hat{z}\left(\partial E_y/\partial x - \partial E_x/\partial y\right) \qquad (5.3.8)$$

Since each cartesian component must equal zero, it follows that $\partial E_x/\partial z = \partial E_z/\partial x$ so both these derivatives are non-zero, as claimed. Note that if the field lines $\overline{E}$ were not curved, then $f_x = 0$ in Figure 5.3.1. But such fields with a gradient $\nabla E_z \neq 0$ would have non-zero curl, which would require current to flow in the insulating region.

The polarization $\overline{P} = \overline{D} - \varepsilon_o\overline{E} = \left(\varepsilon - \varepsilon_o\right)\overline{E}$, as discussed in Section 2.5.3. Thus, in free space, dielectrics with $\varepsilon > \varepsilon_o$ are always drawn into regions with higher field strengths while dielectrics with $\varepsilon < \varepsilon_o$ are always repulsed. The same result arises from energy considerations; the total energy w_e decreases as a dielectric with permittivity ε greater than that of its surrounding ε_o moves into regions having greater field strength $\overline{E}$.

Example 5.3A

What is the Kelvin polarization force density $\overline{F}_p$ [$N\ m^{-3}$] on a dielectric of permittivity $\varepsilon = 3\varepsilon_o$ in a field $\overline{E} = \hat{z}E_o(1 + 5z)$?

Solution: (5.3.7) yields $F_{pz} = \overline{P} \bullet \left(\nabla E_z\right) = \left(\varepsilon - \varepsilon_o\right)\overline{E} \bullet \hat{z}5E_o = 10\varepsilon_o E_o{}^2 \ \left[N\ m^{-3}\right]$.

5.3.3 Kelvin magnetization force density

Magnetic dipoles are induced in permeable materials by magnetic fields. These induced magnetic dipoles arise when the applied magnetic field slightly realigns the randomly oriented pre-existing magnetic dipoles associated with electron spins and electron orbits in atoms. Each such induced magnetic dipole can be modeled as a small current loop, such as the one pictured in Figure 5.3.1(b) in the x-y plane. The collective effect of these induced atomic magnetic dipoles is a permeability μ that differs from μ_o, as discussed further in Section 2.5.4. Prior to realignment of the magnetic dipoles in a magnetizable medium by an externally applied $\overline{H}$, their orientations are generally random so that their effects cancel and can be neglected.

Kelvin magnetization forces on materials result when a non-zero magnetic field gradient causes the Lorentz magnetic forces on the two current centers of each induced magnetic dipole to differ so they no longer cancel, as illustrated in Figure 5.3.1(b). The magnified portion of the figure shows a typical current loop in cross-section where the magnetic Lorentz forces f_{x1} and f_{x2} are unbalanced because $\overline{B}_1$ and $\overline{B}_2$ differ. The magnetic flux density $\overline{B}_1$ acts on the current flowing in the -y direction, and the magnetic field $\overline{B}_2$ acts on the equal and opposite current flowing in the +y direction. The force density $\overline{F}_m$ can be found by summing the net force vectors for every such induced magnetic dipole within a unit volume. This net force density pulls a medium with $\mu > \mu_o$ into the high-field region.

The current loops induced in magnetic materials such as iron and nickel tend to increase the applied magnetic field $\overline{H}$, as illustrated, so that the permeable material in the figure has $\mu > \mu_o$ and experiences a net force that tends to move it toward more intense magnetic fields. That is why magnets attract iron and any *paramagnetic material* that has $\mu > \mu_o$, while repulsing any *diamagnetic material* for which the induced current loops have the opposite polarity so that $\mu < \mu_o$. Although most ordinary materials are either paramagnetic or diamagnetic with $\mu \cong \mu$, only ferromagnetic materials such as iron and nickel have $\mu >> \mu_o$ and are visibly affected by ordinary magnets.

An expression for the Kelvin magnetization force density $\overline{F}_m$ can be derived by calculating the forces on a square current loop of I amperes in the x-y plane, as illustrated. The Lorentz magnetic force on each of the four legs is:

$$\overline{f}_i = \overline{I} \times \mu_o \overline{H} w \quad [N] \tag{5.3.9}$$

where i = 1,2,3,4, and w is the length of each leg. The sum of these four forces is:

$$\begin{aligned} \overline{f} &= Iw^2 \mu_o \left[\left(\hat{y} \times \partial \overline{H} / \partial x \right) - \left(\hat{x} \times \partial \overline{H} / \partial y \right) \right] \quad [N] \\ &= Iw^2 \mu_o \left[-\hat{z} \left(\partial H_x / \partial x + \partial H_y / \partial y \right) + \hat{x} \left(\partial H_z / \partial x \right) + \hat{y} \left(\partial H_z / \partial y \right) \right] \end{aligned} \tag{5.3.10}$$

This expression can be simplified by noting that $\overline{m} = \hat{z} I w^2$ is the magnetic dipole moment of this current loop, and that $\partial H_x/\partial x + \partial H_y/\partial y = - \partial H_z/\partial z$ because $\nabla \bullet \overline{H} = 0$, while $\partial H_z/\partial x = \partial H_x/\partial z$ and $\partial H_z/\partial y = \partial H_y/\partial z$ because $\nabla \times \overline{H} = 0$ in the absence of macroscopic currents. Thus for the geometry of Figure 5.3.1(b), where $\overline{m}$ is in the z direction, the magnetization force of Equation (5.3.10) becomes:

$$\overline{f}_m = \mu_o m_z\left(\hat{z}\partial H_z/\partial z + \hat{x}\partial H_x/\partial z + \hat{y}\partial H_y/\partial z\right) = \mu_o m_z \partial\overline{H}/\partial z \qquad (5.3.11)$$

This expression can be generalized to cases where $\overline{m}$ is in arbitrary directions:

$$\overline{f}_m = \mu_o\left(m_x\partial\overline{H}/\partial x + m_y\partial\overline{H}/\partial y + m_z\partial\overline{H}/\partial z\right) = \mu_o\overline{m}\bullet\nabla\overline{H}\ \ [N] \qquad (5.3.12)$$

where the novel notation $\overline{m}\bullet\nabla\overline{H}$ was defined in (5.3.6).

Equation (5.3.12) then yields the Kelvin magnetization force density $\overline{F}_m = n_3\overline{f}$, where n_3 is the equivalent density of magnetic dipoles [m^{-3}], and the magnetization $\overline{M}$ of the material is $n_3\overline{m}$ [$A\ m^{-1}$]:

$$\overline{F}_m = \mu_o\overline{M}\bullet\nabla\overline{H}\ \ \left[N\ m^{-3}\right] \qquad \text{(Kelvin magnetization force density)} \qquad (5.3.13)$$

Such forces exist even when the applied magnetic field $\overline{H}$ and the magnetization $\overline{M}$ are orthogonal to the field gradient, as illustrated in Figure 5.3.1(b). As in the case of Kelvin polarization forces, this happens in curl-free fields because then a non-zero $\partial H_z/\partial x$ implies a non-zero $\partial H_x/\partial z$ that contributes to $\overline{F}_m$.

5.4 Forces computed using energy methods

5.4.1 Relationship between force and energy

Mechanics teaches that a force f in the z direction pushing an object a distance dz expends energy dw = f dz [J], so:

$$f = dw/dz \qquad \textit{(force/energy equation)} \qquad (5.4.1)$$

Therefore the net force f_{be} applied by the environment to any object in the z direction can be found simply by differentiating the total system energy w with respect to motion of that object in the direction z. The total force vector $\overline{f}_{be}$ is the sum of its x, y, and z components.

Care must be taken, however, to ensure that the total system energy is differentiated, which can include the energy in any connected power supplies, mechanical elements, etc. Care must also be taken to carefully distinguish between forces f_{be} exerted by the environment, and forces f_{oe} exerted by objects on their environment; otherwise sign errors are readily introduced. This simple powerful approach to finding forces is illustrated in Section 5.4.2 for electrostatic forces and in Section 5.6 for photonic forces. The energy approach to calculating magnetic forces uses (5.4.1) in a straightforward way, but examples are postponed to Chapter 6 when magnetic fields in structures will be better understood.

Example 5.4A

A certain perfectly conducting electromagnet carrying one ampere exerts an attractive 100-N force f on a piece of iron while it moves away from the magnet at velocity v = 1 [m s^{-1}]. What voltage V is induced across the terminals of the electromagnet as a result of this velocity v? Is this voltage V positive or negative on that terminal where the current enters the magnet? Use conservation of power.

Solution: Conservation of power requires fv = VI, so V = fv/I = 100×1/1 = 100 volts. The voltage is negative because the magnet is acting as a generator since the motion of the iron is opposite to the magnetic force acting on it.

5.4.2 Electrostatic forces on conductors and dielectrics

The energy method easily yields the force f_{be} needed to separate in the z direction the two isolated capacitor plates oppositely charged with Q in vacuum and illustrated in Figure 5.2.1(a). Since the plates are attracted to one another, separating them does work and increases the stored energy w. The force needed to hold the plates apart is easily found using the force/energy equation (5.4.1):

$$f_{be} = dw/dz = d\left(Q^2 s/2\varepsilon_o A\right)/ds = Q^2/2\varepsilon_o A \quad [\mathrm{N}] \tag{5.4.2}$$

where the plate separation is s and the plate area is A. The electric energy w_e stored in a capacitor C is $CV^2/2 = Q^2/2C = Q^2 s/2\varepsilon A$, where Q = CV and C = εA/s, as shown in Section 3.1.3. Here we assumed $\varepsilon = \varepsilon_o$.

The derivative in (5.4.2) was easy to evaluate because Q remains constant as the disconnected plates are forced apart. It would be incorrect to use $w = CV^2/2$ when differentiating (5.4.2) unless we recognize that V increases as the plates separate because V = Q/C when C decreases. It is easier to express energy in terms of parameters that remain constant as z changes.

We can put (5.4.2) in the more familiar form (5.2.4) for the electric pressure P_e pushing on a conductor by noting that the force f_{be} needed to separate the plates is the same as the electric force attracting the oppositely charged plates. The force f_{be} thus balances the electric pressure on the same plates and $P_e = -f_{be}/A$. Since $Q = \varepsilon_o EA$ here we find:

$$P_e = -Q^2/2\varepsilon_o A^2 = -\varepsilon_o E^2/2 \quad \left[\mathrm{Nm^{-2}}\right] \tag{5.4.3}$$

This static attractive pressure of electric fields remains the same if the plates are connected to a battery of voltage V instead of being isolated; the Lorentz forces are the same in both cases. A more awkward way to calculate the same force (5.4.2) is to assume (unnecessarily) that a battery is connected and that V remains constant as s changes. In this case Q must vary with dz, and dQ flows into the battery, increasing its energy by VdQ. Since dw in the force/energy expression (5.4.2) is the change in total system energy, the changes in both battery and electric field energy must be calculated to yield the correct energy; an example with a battery begins later with (5.4.5). As illustrated above, this complexity can be avoided by carefully restating the problem without the source, and by expressing w in terms of electrical variables (Q here) that do not vary with position (s here).

The power of the energy method (5.4.1) is much more evident when calculating the force $\overline{f}$ needed to pull two capacitor plates apart laterally, as illustrated in Figure 5.4.1(a). To use the Lorentz force law directly would require knowledge of the lateral components of $\overline{E}$ responsible for the lateral forces, but they are not readily determined. Since energy derivatives can often be computed accurately and easily (provided the fringing fields are relatively small), that is often the preferred method for computing electric and magnetic forces.

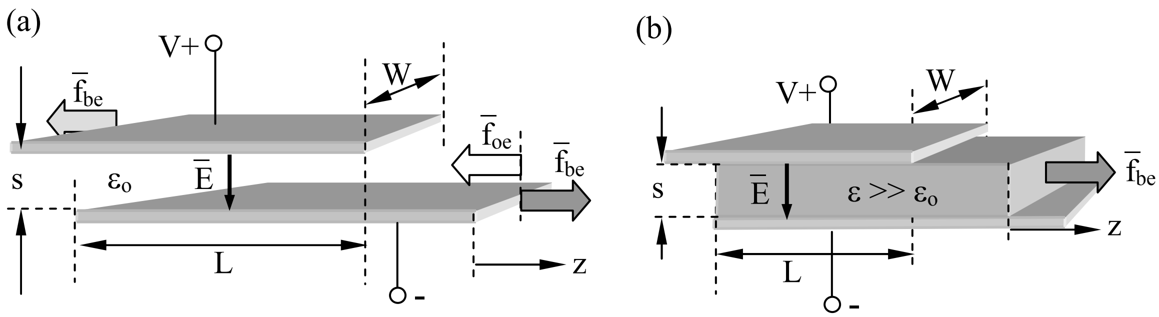

Figure 5.4.1 Capacitor plates and dielectrics being separated laterally.

The force/energy equation (5.4.1) can be expressed in terms of the area A = WL of the capacitor. Because L decreases as z increases, the sign of the derivative with respect to the plate overlap L is negative, and the force exerted on the plates by the environment is:

$$f_{be} = dw/dz = -d\left(Q^2 s/2\varepsilon_o WL\right)/dL = Q^2 s/2\varepsilon_o WL^2 \ \ [\mathrm{N}] \tag{5.4.4}$$

where dz = dL and $w_e = Q^2 s/2WL$. We again assumed that the plates were isolated in space so Q was constant, but the same force results when the plates are attached to a battery; in both cases the Lorentz forces arise from the very same charges so the two forces must be identical.

For purposes of illustration, let's solve the force/energy equation (5.4.1) for the same problem of Figure 5.4.1 the more difficult way by including the increase in battery energy as z increases. The incremental work $f_{be}dz$ involved in pulling the plates apart a distance dz is:

$$f_{be} = dw_T/dz = -d\left(\varepsilon_o WLV^2/2s\right)/dL - VdQ/dz \tag{5.4.5}$$

where w_T is the total energy and the two terms on the right-hand side of (5.4.5) reflect the energy changes in the capacitor and battery respectively. The first negative sign in (5.4.5) arises because the overlap distance L decreases as z increases, and the second negative sign arises because the battery energy increases as Q decreases.

Since only L and Q vary with L, where $Q = CV = \varepsilon_o WLV/s$, (5.4.5) becomes:

$$f_{be} = -\varepsilon_o WV^2/2s + \varepsilon_o WV^2/s = \varepsilon_o WV^2/2s \quad [N] \tag{5.4.6}$$

where the sign of the second term ($\varepsilon_o WV^2/s$) reverses because Q decreases as z increases. This result when including the battery is the same as (5.4.4) without the battery, which can be seen by using $V = Q/C$ and $C = \varepsilon_o WL/s$:

$$f_{be} = \varepsilon_o WV^2/2s = Q^2 s/2\varepsilon_o WL^2 \quad [N] \tag{5.4.7}$$

If the space between and surrounding the conducting plates were filled with a fluid having $\varepsilon > \varepsilon_o$, then for fixed V both the stored electric energy w_e and dw_e/dz, together with the force f_{be}, would obviously be increased by a factor of $\varepsilon/\varepsilon_o$ so that in this case the lateral force f_{be} would equal $\varepsilon WV^2/2s$.

Note that approximately the same force f_{be} is required to separate laterally two capacitor plates, one of which is coated with a dielectric having permittivity ε, as illustrated in Figure 5.4.1(b), because the force/energy equation (5.4.4) is largely unchanged except that $\varepsilon_o \rightarrow \varepsilon$:

$$f_{be} = dw/dz = -d\left(Q^2 s/2\varepsilon WL\right)/dL = Q^2 s/2\varepsilon WL^2 \, [N] \tag{5.4.8}$$

5.5 Electric and magnetic pressure

5.5.1 Electromagnetic pressures acting on conductors

Forces on materials can be computed in several different ways, all of which can be derived using Maxwell's equations and the Lorentz force law. The pressure method for computing forces arising from static fields is useful because it expresses prior results in ways that are easy to evaluate and remember, and that have physical significance. The method simply notes that the electromagnetic force density (pressure) acting on the interface between two materials equals the difference in the electromagnetic energy densities on the two sides of the interface. Both energy density [J m^{-3}] and pressure [N m^{-2}] have identical units because [J] = [N m].

For example, both the Lorentz force law and the energy method yield the same expression, (5.2.4) and (5.4.3) respectively, for the *electric pressure* P_e due to a static electric field E pushing on a conductor:

$$P_e = -\varepsilon_o E^2/2 \, \left[N\ m^{-2}\right] \qquad \text{(electric pressure on conductors)} \tag{5.5.1}$$

The Lorentz force law yields a similar expression (5.2.13) for the *magnetic pressure* pushing on a conductor:

$$P_m = \mu_o H^2/2 \ \left[\mathrm{N\ m^{-2}}\right] \qquad \text{(magnetic pressure on conductors)} \qquad (5.5.2)$$

Thus motor and actuator forces are limited principally by the ability of material systems to sustain large static fields without breaking down in some way. Because large magnetic systems can sustain larger energy densities than comparable systems based on electric fields, essentially all large motors, generators, and actuators are magnetic. Only for devices with gaps on the order of a micron or less is the electrical breakdown field strength sufficiently high that electrostatic and magnetic motors compete more evenly with respect to power density, as discussed in Section 6.2.5.

5.5.2 Electromagnetic pressures acting on permeable and dielectric media

The Kelvin polarization and magnetization force densities, (5.3.7) and (5.3.13) respectively, can also be expressed in terms of energy densities and pressures. First we recall that $\overline{D} = \varepsilon\overline{E} = \varepsilon_o\overline{E} + \overline{P}$, so $\overline{P} = (\varepsilon - \varepsilon_o)\overline{E}$. Then it follows from (5.3.7) that the Kelvin polarization force density is:

$$\overline{F}_p = \overline{P} \bullet \nabla\overline{E} = (\varepsilon - \varepsilon_o)\overline{E} \bullet \nabla\overline{E} \ \left[\mathrm{N\ m^{-3}}\right] \qquad (5.5.3)$$

The special operator [$\bullet\nabla$] is defined in (5.3.6) and explained in (5.5.4). The x component of force density for a curl-free electric field $\overline{E}$ is:

$$F_{px} = \overline{P} \bullet (\nabla E_x) = (\varepsilon - \varepsilon_o)\overline{E} \bullet \nabla E_x = (\varepsilon - \varepsilon_o)\left(E_x\partial/\partial x + E_y\partial/\partial y + E_z\partial/\partial z\right)E_x \qquad (5.5.4)$$

$$= (\varepsilon - \varepsilon_o)\left(E_x\partial E_x/\partial x + E_y\partial E_y/\partial x + E_z\partial E_z/\partial x\right) \qquad (5.5.5)$$

$$= (\varepsilon - \varepsilon_o)\left(E_x\partial E_x{}^2/\partial x + E_y\partial E_y{}^2/\partial x + E_z\partial E_z{}^2/\partial x\right)/2 = (\varepsilon - \varepsilon_o)\left(\partial|\overline{E}|^2/\partial x\right)/2 \qquad (5.5.6)$$

In obtaining (5.5.5) we have used (5.3.8) for a curl-free electric field, for which $\partial E_x/\partial y = \partial E_y/\partial x$ and $\partial E_x/\partial z = \partial E_z/\partial x$.

Equations similar to (5.5.6) can be derived for the y and z components of the force density, which then add:

$$\overline{F}_p = (\varepsilon - \varepsilon_o)\nabla|\overline{E}|^2/2 \ \left[\mathrm{N\ m^{-3}}\right] \qquad \text{(\textit{Kelvin polarization force density})} \qquad (5.5.7)$$

A similar derivation applies to the Kelvin magnetization force density $\overline{F}_m$. We begin by recalling $\overline{B} = \mu_o\overline{M} \bullet \nabla\overline{H}$, so $\overline{M} = \left[(\mu/\mu_o) - 1\right]\overline{H}$. Then it follows from (5.3.13) that the Kelvin magnetization force density is:

$$\overline{F}_m = \mu_o \overline{M} \bullet \nabla \overline{H} = (\mu - \mu_o) \overline{H} \bullet \nabla \overline{H} \;\left[\mathrm{N\ m^{-3}}\right] \tag{5.5.8}$$

Repeating the steps of (5.5.4–7) yields for curl-free magnetic fields the parallel result:

$$\overline{F}_m = (\mu - \mu_o) \nabla |\overline{H}|^2 / 2 \;\left[\mathrm{N\ m^{-3}}\right] \quad (\textit{Kelvin magnetization force density}) \tag{5.5.9}$$

Note that these force density expressions depend only on the field magnitudes $|\overline{E}|$ and $|\overline{H}|$, not on field directions.

Two examples treated in Chapter 6 using energy methods suggest the utility of simple pressure equations. Figure 6.2.4 shows a parallel-plate capacitor with a dielectric slab that fits snugly between the plates but that is only partially inserted in the z direction a distance D that is much less than the length L of both the slab and the capacitor plates. The electric field between the plates is $\overline{E}$, both inside and outside the dielectric slab. The total force on the dielectric slab is the integral of the Kelvin polarization force density (5.5.7) over the volume V of the slab, where V = LA and A is the area of the endface of the slab. We find from (5.5.7) that $\overline{F}_p$ is in the $\hat{z}$ direction and is non-zero only near the end of the capacitor plates where z = 0:

$$f_z = A \int_0^D F_{pz}\, dz = A\left[(\varepsilon - \varepsilon_o)/2\right] \int_0^D \left(d|\overline{E}|^2 / dz\right) dz = A(\varepsilon - \varepsilon_o)|\overline{E}|^2 / 2 \;\;[\mathrm{N}] \tag{5.5.10}$$

The integral is evaluated between the limit z = 0 where E ≅ 0 outside the capacitor plates, and the maximum value z = D where the electric field between the plates is $\overline{E}$. Thus the pressure method yields the total force f_z on the dielectric slab; it is the area A of the end of the slab, times the electric pressure $(\varepsilon - \varepsilon_o)|\overline{E}|^2/2$ [N m^{-2}] at the end of the slab that is pulling the slab further between the plates. This pressure is zero at the other end of the slab because $\overline{E} \cong 0$ there. This pressure is the same as will be found in (6.2.21) using energy methods.

The second example is illustrated in Figure 6.4.1, where a snugly fitting cylindrical iron slug of area A has been pulled a distance D into a solenoidal coil that produces an axial magnetic field H. As in the case of the dielectric slab, one end of the slug protrudes sufficiently far from the coil that H at that end is approximately zero. The force pulling on the slug is easily found from (5.5.9):

$$f_z = A \int_0^D F_{mz}\, dz = A\left[(\mu - \mu_o)/2\right] \int_0^D \left(d|\overline{H}|^2 / dz\right) dz = A(\mu - \mu_o)|\overline{H}|^2 / 2 \;\;[\mathrm{N}] \tag{5.5.11}$$

This is more exact than the answer found in (6.4.10), where the μ_o term was omitted in (6.4.10) when the energy stored in the air was neglected.

To summarize, the static electromagnetic pressure [N m^{-2}] acting on a material interface with either free space or mobile liquids or gases is the difference between the two electromagnetic energy densities [J m^{-3}] on either side of that interface, provided that the relevant $\overline{E}$ and $\overline{H}$ are curl-free. In the case of dielectric or magnetic media, the pressure on the material is directed away from the greater energy density. In the case of conductors, external magnetic fields press on them while electric fields pull; the energy density inside the conductor is zero in both cases because $\overline{E}$ and $\overline{H}$ are presumed to be zero there.

Note that the pressure method for calculating forces on interfaces is numerically correct even when the true physical locus of the force may lie elsewhere. For example, the Kelvin polarization forces for a dielectric slab being pulled into a capacitor are concentrated at the edge of the capacitor plates at z = 0 in Figure 6.2.4, which is physically correct, whereas the pressure method implies incorrectly that the force on the slab is concentrated at its end between the plate where z = D. The energy method does not address this issue.

Example 5.5A

At what radius r from a 1-MV high voltage line does the electric force acting on a dust particle having $\varepsilon = 10\varepsilon_o$ exceed the gravitational force if its density ρ is 1 gram/cm^3? Assume the electric field around the line is the same as between concentric cylinders having radii a = 1 cm and b = 10 m.

Solution: The Kelvin polarization force density (5.5.7) can be integrated over the volume v of the particle and equated to the gravitational force $f_g = \rho v g = \sim 10^{-3} v 10$ [N]. (5.5.7) yields the total Kelvin force: $\overline{f}_K = v(\varepsilon - \varepsilon_o)\nabla|\overline{E}|^2/2$ where $\overline{E}(r) = \hat{r}\,V/[r\ln(b/a)]$ [V m^{-1}]. $\nabla|\overline{E}|^2 = [V/\ln(b/a)]^2\nabla r^{-2} = -2\hat{r}\,[V/\ln(b/a)]^2 r^{-3}$, where the gradient here, $\nabla = \hat{r}\,\partial/\partial r$, was computed using cylindrical coordinates (see Appendix C). Thus $f_g = |\overline{f}_K|$ becomes $10^{-2}v = v9\varepsilon_o\,[V/\ln(b/a)]^2 r^{-3}$, so $r = \{900\varepsilon_o[V/\ln(b/a)]^2\}^{1/3} = \{900\times 8.85\times 10^{-12}[10^6/\ln(1000)]^2\}^{1/3} = 5.5$ meters, independent of the size of the particle. Thus high voltage lines make excellent dust catchers for dielectric particles.

5.6 *Photonic forces*

Photonic forces arise whenever electromagnetic waves are absorbed or reflected by objects, and can be found using either wave or photon paradigms. Section 5.2.2 derived the magnetic pressure $\overline{P}_m$ (5.2.13) applied by a surface magnetic field $H_s(t)$ that is parallel to a flat perfect conductor in the x-y plane:

$$\overline{P}_m = \hat{z}\mu_o H_s^2/2\ \left[\mathrm{Nm}^{-2}\right] \qquad \text{(magnetic pressure on perfect conductor)} \qquad (5.6.1)$$

Thus this instantaneous magnetic pressure perpendicular to the conductor surface equals the adjacent magnetic energy density ([N m^{-2}] = [J m^{-3}]).

In the sinusoidal steady state the time average pressure is half the peak instantaneous value given by (5.6.1), where $H_s(t) = H_s \cos\omega t$. This average pressure on a perfectly reflecting conductor can also be expressed in terms of the time-average Poynting vector $\langle \overline{S}(t) \rangle$ of an incident wave characterized by $H_+ \cos \omega t$:

$$\langle \overline{P}_m(t) \rangle = \hat{z}\mu_o \langle H_s^2(t) \rangle / 2 = 2\langle \overline{S}(t) \rangle / c \quad \left[Nm^{-2} \right] \tag{5.6.2}$$

where $H_s = 2H_+$ and $\langle S(t) \rangle = \eta_o H_+^2/2$; the impedance of free space $\eta_o = \mu_o/c$.

It is now easy to relate $\langle S(t) \rangle$ to the photon momentum flux, which also yields pressure. We recall[17] that:

$$\textit{photon momentum}\ M = hf/c \quad \left[Nm\ s^{-1} \right] \tag{5.6.3}$$

The momentum transferred to a mirror upon perfect reflection of a single photon at normal incidence is therefore 2hf/c.

We recall from mechanics that the force f required to change momentum mv is:

$$f = d(mv)/dt \quad [N] \tag{5.6.4}$$

so that the total *radiation pressure* on a perfect mirror reflecting directly backwards n photons [$s^{-1}m^{-2}$] is:

$$\langle P_r \rangle = n2hf/c = 2\langle S(t) \rangle / c \ \left[Nm^{-2} \right] \qquad \text{(radiation pressure on a mirror)} \tag{5.6.5}$$

consistent with (5.6.2). Thus we have shown that both the Lorentz force method and the photonic force method yield the same pressure on perfectly reflecting mirrors; $P_m = P_r$. The factor of two in (5.6.5) arises because photon momentum is not zeroed but reversed by a mirror. If these photons were absorbed rather than reflected, the rate of momentum transfer to the absorber would be halved. In general if the incident and normally reflected power densities are $\langle S_1 \rangle$ and $\langle S_2 \rangle$, respectively, then the average radiation pressure on the mirror is:

$$\langle P \rangle = \langle S_1 + S_2 \rangle / c \tag{5.6.6}$$

If the photons are incident at an angle, the momentum transfer is reduced by the cosine of the angle of incidence and reflection. And if the mirror is partially transparent, the momentum transfer is reduced by that fraction of the photon momentum that passes through unaltered.

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[17] A crude plausibility argument for (5.6.3) is the following. The energy of a photon is hf [J], half being magnetic and half being electric. We have seen in (5.2.1) and (5.2.13) that only the magnetic fields contribute to the Lorentz force on a normal reflecting conductor for which both $E_\perp$ and $H_\perp = 0$, so we might notionally associate hf/2 with the "kinetic energy of a photon", where kinetic energy is linked to momentum. If photons had mass m, this notional kinetic energy hf/2 would equal $mc^2/2$, and the notional associated momentum mc of a photon would then equal hf/c, its actual value.

Consider the simple example of a reflective *solar sail* blown by radiation pressure across the solar system, sailing from planet to planet. At earth the *solar radiation* intensity is ~1400 W/m^2, so (5.6.6) yields, for example, the total force f on a sail of projected area A intercepting one square kilometer of radiation:

$$f = A\langle P\rangle = A2\langle S(t)\rangle/c \leq 10^6 \times 2 \times 1400\big/\left(3\times 10^8\right) \cong 9[N] \qquad (5.6.7)$$

A sail this size one micron thick and having the density of water would have a mass m of 1000 kg. Since the sail velocity v = at = (f/m)t, where a is acceleration and t is time, it follows that after one year the accumulated velocity of a sail facing such constant pressure in vacuum could be as much as $(9/1000)3\times10^7 \cong 3\times10^5$ ms^{-1} = c/1000. Of course the solar photon pressure declines as the square of the solar distance, and solar gravity would also act on such sails.

Example 5.6A

What force F [N] is exerted on a 3-watt flashlight ($\lambda \cong 0.5$ microns) as a result of the exiting photons?

Solution: E = hf and power P = Nhf = 3 watts, where N is the number of photons per second. The force F = Nhf/c, where hf/c is the momentum of a single photon, and N = 3/hf here. So F = 3/c = 10^{-8} Newtons. A Newton approximates the gravitational force on the quarter-pound package of fig newtons. This force pushes the flashlight in the direction opposite to that of the light beam.

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