Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Fall 2010)

Recitation 10
October 12, 2010

- **Question 1.** The two parts of this question are about identities for a probabilistic model with sample space Ω, events A and B, and discrete random variable X. Any time conditioning on an event is indicated, the event has positive probability. An identity is *true* when it holds without any additional restrictions; it is *false* when there is any counterexample.

  1.1. Which **one** of the following statements is **true**?
  - (a) $\mathbf{P}(A \cap B)$ may be larger than $\mathbf{P}(A)$.
  - (b) The variance of X may be larger than the variance of $2X$.
  - (c) If $A^c \cap B^c = \emptyset$, then $\mathbf{P}(A \cup B) = 1$.
  - (d) If $A^c \cap B^c = \emptyset$, then $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$.
  - (e) If $\mathbf{P}(A) > 1/2$ and $\mathbf{P}(B) > 1/2$, then $\mathbf{P}(A \cup B) = 1$.

  1.2. Which **one** of the following statements is **true**?
  - (a) If $\mathbf{E}[X] = 0$, then $\mathbf{P}(X > 0) = \mathbf{P}(X < 0)$.
  - (b) $\mathbf{P}(A) = \mathbf{P}(A \mid B) + \mathbf{P}(A \mid B^c)$
  - (c) $\mathbf{P}(B \mid A) + \mathbf{P}(B \mid A^c) = 1$
  - (d) $\mathbf{P}(B \mid A) + \mathbf{P}(B^c \mid A^c) = 1$
  - (e) $\mathbf{P}(B \mid A) + \mathbf{P}(B^c \mid A) = 1$

- **Question 2.** Provide **clear reasoning**; partial credit is possible

  Heather and Taylor play a game using independent tosses of an unfair coin. A head comes up on any toss with probability p, where $0 < p < 1$. The coin is tossed repeatedly until either the second time head comes up, in which case Heather wins; or the second time tail comes up, in which case Taylor wins. Note that a full game involves 2 or 3 tosses.

  2.1. Consider a probabilistic model for the game in which the outcomes are the sequences of heads and tails in a full game. Provide a list of the outcomes and their probabilities of occurring.

  2.2. What is the probability that Heather wins the game?

  2.3. What is the conditional probability that Heather wins the game given that head comes up on the first toss?

  2.4. What is the conditional probability that head comes up on the first toss given that Heather wins the game?

- **Question 3.** Provide **clear reasoning**; partial credit is possible

A casino game using a **fair** 4-sided die (with labels 1, 2, 3, and 4) is offered in which a **basic game** has 1 or 2 die rolls:

- If the first roll is a 1, 2, or 3, the player wins the amount of the die roll, in dollars, and the game is over.
- If the first roll is a 4, the player wins \$2 and the amount of a second ("bonus") die roll in dollars.

Let X be the payoff in dollars of the basic game.

3.1. Find the PMF of X, $p_X(x)$.

3.2. Find $\mathbf{E}[X]$.

3.3. Find the conditional PMF of the result of the first die roll given that $X = 3$. (Use a reasonable notation that you define explicitly.)

3.4. Now consider an **extended game** that can have any number of bonus rolls. Specifically:

   * Any roll of a 1, 2, or 3 results in the player winning the amount of the die roll, in dollars, and the termination of the game.
   * Any roll of a 4 results in the player winning \$2 and continuation of the game.

   Let Y denote the payoff in dollars of the extended game. Find $\mathbf{E}[Y]$.