Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Fall 2010)

Recitation 12 Solutions
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1.

$$\begin{aligned}
\rho(aX+b,Y) &= \frac{\text{cov}(aX+b,Y)}{\sqrt{\text{var}(aX+b)(\text{var}(Y))}} \\
&= \frac{\mathbf{E}[(aX+b-\mathbf{E}[aX+b])(Y-\mathbf{E}[Y])]}{\sqrt{a^2\text{var}(X)\text{var}(Y)}} \\
&= \frac{\mathbf{E}[(aX+b-a\mathbf{E}[X]-b)(Y-\mathbf{E}[Y])]}{a\sqrt{\text{var}(X)\text{var}(Y)}} \\
&= \frac{a\mathbf{E}[(X-\mathbf{E}[X])(Y-\mathbf{E}[Y])]}{a\sqrt{\text{var}(X)\text{var}(Y)}} \\
&= \frac{\text{cov}(X,Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} \\
&= \rho(X,Y)
\end{aligned}$$

As an example where this property of the correlation coefficient is relevant, consider the homework and exam scores of students in a class. We expect the homework and exam scores to be positively correlated and thus have a positive correlation coefficient. Note that, in this example, the above property will mean that the correlation coefficient will not change whether the exam is out of 105 points, 10 points, or any other number of points.

2.

(a) When $z \geq 0$:

$$\begin{aligned}
F_Z(z) = \mathbf{P}(X-Y\leq z) &= \mathbf{P}(X \leq Y+z) \\
&= \int_0^\infty \int_0^{y+z} f_{X,Y}(x,y')dxdy \\
&= \int_0^\infty \lambda e^{-\lambda y} \int_0^{y+z} \lambda e^{-\lambda x} dxdy \\
&= \lambda^{\lambda y}\left(1-e^{-\lambda(y+z)}\right) dy \\
&= 1+\frac{e^{-\lambda z}}{2}e^{-2\lambda y}\Big|_{y=0}^{y=\infty} \\
&= 1-\frac{1}{2}e^{-\lambda z} \qquad z \geq 0
\end{aligned}$$

When $z < 0$:

$$\begin{aligned}
\mathbf{P}(X \leq Y + z) &= \int_0^\infty \int_0^{y+z} f_{X,Y}(x,y)dxdy \\
&= \int_0^\infty \lambda e^{-\lambda x} \int_{x-z}^\infty \lambda e^{-\lambda y} dydx \\
&= \int_0^\infty \lambda e^{-\lambda x} e^{-\lambda(x-z)} dx \\
&= e^{\lambda z} \int_0^\infty \lambda e^{-2\lambda x} dx \\
&= \frac{1}{2} e^{\lambda z} \qquad z \leq 0
\end{aligned}$$

$$F_Z(z) = \begin{cases} 1 - \frac{1}{2}e^{-\lambda z} & z \geq 0 \\ \frac{1}{2}e^{\lambda z} & z < 0 \end{cases}$$

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \begin{cases} \frac{\lambda}{2}e^{-\lambda z} & z \geq 0 \\ \frac{\lambda}{2}e^{\lambda z} & z < 0 \end{cases}$$

Hence,

$$f_Z(z) = \frac{\lambda}{2} e^{\lambda|z|}$$

(b) Solving using the total probability theorem, we have:

$$\begin{aligned}
f_Z(z) &= \int_{-\infty}^\infty f_X(x) f_{Z|X}(z|x)dx \\
&= \int_{-\infty}^\infty f_X(x) f_{Y|X}(x-z|x)dx \\
&= \int_{-\infty}^\infty f_X(x) f_Y(x-z)dx
\end{aligned}$$

First when $z < 0$, we have:

$$\begin{aligned}
\int_{-\infty}^\infty f_X(x) f_Y(x-z)dx &= \int_0^\infty \lambda e^{-\lambda x} \lambda e^{-\lambda(x-z)} dx \\
&= \lambda e^{\lambda z} \int_0^\infty \lambda e^{-2\lambda x} dx \\
&= \frac{\lambda}{2} e^{\lambda z}
\end{aligned}$$

Then, when $z \geq 0$ we have:

$$\begin{aligned}
\int_{-\infty}^{\infty} f_X(x) f_Y(x-z) dx &= \int_{z}^{\infty} \lambda e^{-\lambda x} \lambda e^{-\lambda(x-z)} dx \\
&= \lambda e^{\lambda z} \int_{z}^{\infty} \lambda e^{-2\lambda x} dx \\
&= \frac{\lambda}{2} e^{\lambda z} e^{-2\lambda z} \\
&= \frac{\lambda}{2} e^{-\lambda z} \\
f_Z(z) &= \frac{\lambda}{2} e^{-\lambda |z|} \qquad \forall z
\end{aligned}$$

3. (a) We have $X = R\cos(\Theta)$ and $Y = R\sin(\Theta)$. Recall that in polar coordinates, the differential area is $dA = dxdy = rdrd\theta$. So

$$\begin{aligned}
F_R(r) = \mathbf{P}(R \leq r) &= \int_0^r \int_0^{2\pi} f_X(r'\cos\theta) f_Y(r'\sin\theta) d\theta\, r' dr' \\
&= \int_0^r \int_0^{2\pi} \frac{1}{2\pi} e^{-(r')^2/2} d\theta\, r' dr' \\
&= \int_0^r r' e^{-(r')^2/2} dr' \int_0^{2\pi} \frac{d\theta}{2\pi} \\
&= \int_0^{r^2/2} e^{-u} du \qquad (u = (r')^2/2) \\
F_R(r) &= \begin{cases} 1 - e^{-r^2/2} & r \geq 0 \\ 0 & r < 0 \end{cases}
\end{aligned}$$

$$\begin{aligned}
f_R(r) = \frac{d}{dr} F_R(r) &= (-1/2)(2r)(-e^{-r^2/2}) \\
&= re^{-r^2/2}, \qquad r \geq 0
\end{aligned}$$

(b)

$$\begin{aligned}
F_\Theta(\theta) = \mathbf{P}(\Theta \le \theta) &= \int_0^\theta \int_0^\infty f_X(r\cos\theta') f_Y(r\sin\theta') r dr\, d\theta' \\
&= \int_0^\theta \int_0^\infty \frac{1}{2\pi} e^{-r^2/2} r dr\, d\theta' \\
&= \int_0^\infty r e^{-r^2/2} dr \int_0^\theta \frac{d\theta'}{2\pi} \\
&= \frac{\theta}{2\pi} \int_0^\infty e^{-u} du \qquad (u = r^2/2) \\
&= \frac{\theta}{2\pi} \left(-e^{-u}\right)\Big|_0^\infty = \frac{\theta}{2\pi} \qquad 0 \le \theta \le 2\pi
\end{aligned}$$

$$F_\Theta(\theta) = \begin{cases} 0 & \theta < 0 \\ \frac{\theta}{2\pi} & 0 \le \theta \le 2\pi \\ 1 & \theta \ge 2\pi \end{cases}$$

$$f_\Theta(\theta) = \frac{d}{d\theta} F_\Theta(\theta) = \frac{1}{2\pi} \qquad 0 \le \theta \le 2\pi$$

(c)

$$\begin{aligned}
F_{R,\Theta}(r,\theta) = P(R \le r, \Theta \le \theta) &= \int_0^\theta \int_0^r \frac{1}{2\pi} r' e^{-(r')^2/2} dr'\, d\theta' \\
&= \int_0^\theta \int_0^{r^2/2} \frac{1}{2\pi} e^{-u} du\, d\theta' \qquad (u = (r')^2/2) \\
&= \int_0^\theta \frac{1}{2\pi} \left(1 - e^{-r^2/2}\right) d\theta' \\
&= \frac{\theta}{2\pi} \left(1 - e^{-r^2/2}\right) \qquad r \ge 0, \quad \theta > 2\pi
\end{aligned}$$

$$F_{R,\Theta}(r,\theta) = \begin{cases} \frac{\theta}{2\pi}\left(1 - e^{-r^2/2}\right) & r \ge 0, \quad 0 \le \theta \le 2\pi \\ 1 - e^{-r^2/2} & r \ge 0, \quad \theta > 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$$f_{R,\Theta}(r,\theta) = \frac{\partial}{\partial r}\frac{\partial}{\partial \theta} F_{R,\Theta}(r,\theta) = \frac{1}{2\pi} r e^{-r^2/2} \qquad r \ge 0, \quad 0 \le \theta \le 2\pi$$

Note: The PDF of R^2 is exponentially distributed with parameter $\lambda = 1/2$. This is a very convenient way to generate normal random variables from independent uniform and exponential random variables. We can generate an arbitrary random variable X with CDF F_X by first generating a uniform random variable and then passing the samples from the uniform distribution through the function F_X^{-1}. But since we don't have a closed-form expression for the CDF of a normal random variable, this method doesn't work. However, we do have a closed-form expression for the exponential distribution. Therefore, we can generate an exponential distribution with parameter 1/2 and we can generate a uniform distribution in $[0, 2\pi]$, and with these two distributions we can generate standard normal distributions.

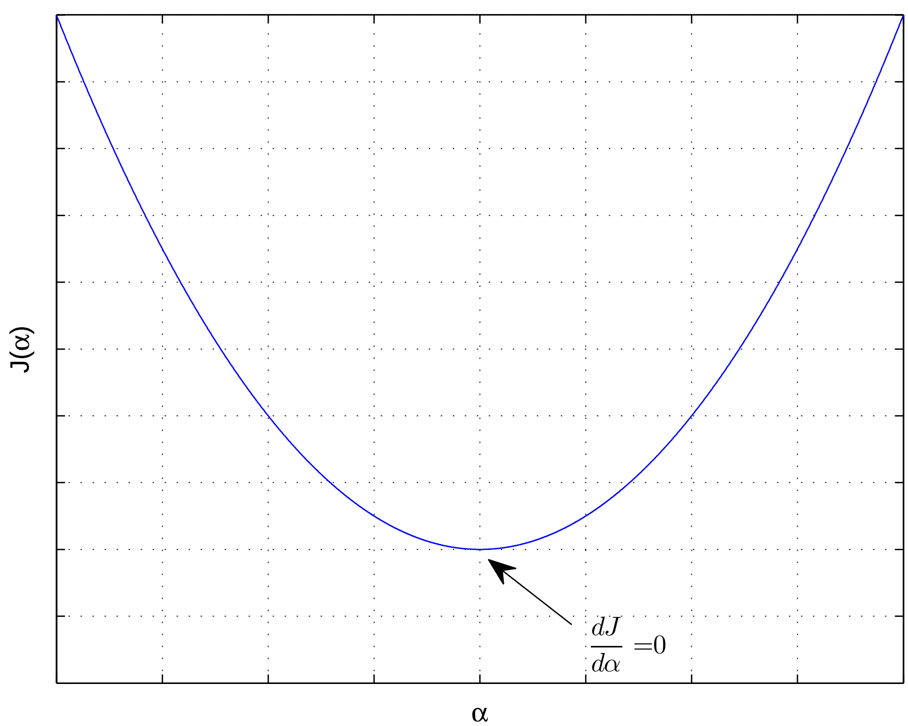
4. Problem 4.20, page 250 in text. See text for the proof.

An alternative proof is given below:

Consider the problem of picking a parameter α to minimize the expected squared difference between two random variables X and Y. Consider

$$J(\alpha) = \mathbf{E}\left[(X - \alpha Y)^2\right]$$

with $Y \neq 0$. We start with a variational calculation to find α that minimizes $J(\alpha)$. The value of α which minimizes $J(\alpha)$ is found by setting the first derivative of $J(\alpha)$ to zero (since, for $Y \neq 0$, $\frac{d^2}{d\alpha^2}J(\alpha) = 2\mathbf{E}[Y^2] > 0$).



$$\frac{d}{d\alpha}J(\alpha) = \frac{d}{d\alpha}\left(\mathbf{E}[X^2] - 2\alpha\mathbf{E}[XY] + \alpha^2\mathbf{E}[Y^2]\right) = 0$$

$$\rightarrow \alpha = \frac{\mathbf{E}[XY]}{\mathbf{E}[Y^2]} \text{ minimizes } J(\alpha).$$

Then

$$\begin{aligned} J\left(\frac{\mathbf{E}[XY]}{\mathbf{E}[Y^2]}\right) &= \mathbf{E}\left[\left(X - \frac{\mathbf{E}[XY]}{\mathbf{E}[Y^2]}Y\right)^2\right] \geq 0 \\ &= \mathbf{E}[X^2] - 2\frac{(\mathbf{E}[XY])^2}{\mathbf{E}[Y^2]} + \frac{(\mathbf{E}[XY])^2\mathbf{E}[Y^2]}{(\mathbf{E}[Y^2])^2} \\ &= \mathbf{E}[X^2] - \frac{(\mathbf{E}[XY])^2}{\mathbf{E}[Y^2]} \geq 0 \end{aligned}$$

Rearranging this expression gives the Schwarz inequality for expected values:

$$\mathbf{E}[X^2]\mathbf{E}[Y^2] \geq (\mathbf{E}[XY])^2$$

Note that in the above derivation, we assumed $Y \neq 0$ so that $\mathbf{E}[Y^2] > 0$. If we assume $Y = 0$ then the Schwarz inequality will hold with equality since then $\mathbf{E}[XY] = 0$ and $\mathbf{E}[Y^2] = 0$.