Problem Set 4: Solutions
Due: March 8, 2006

1. (a) Use the total probability theorem by conditioning on the number of questions that Professor Right has to answer. Let A be the event that she gives all wrong answers in a given lecture, let B_1 be the event that she gets one question in a given lecture, and let B_2 be the event that she gets two questions in a given lecture. Then

$$\mathbf{P}(A) = \mathbf{P}(A|B_1)\mathbf{P}(B_1) + \mathbf{P}(A|B_2)\mathbf{P}(B_2).$$

From the problem statement, she is equally likely to get one or two questions in a given lecture, so $\mathbf{P}(B_1) = \mathbf{P}(B_2) = \frac{1}{2}$. Also, from the problem statement, $\mathbf{P}(A|B_1) = \frac{1}{4}$, and, because of independence, $\mathbf{P}(A|B_2) = (\frac{1}{4})^2 = \frac{1}{16}$. Thus we have

$$\mathbf{P}(A) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{16} \cdot \frac{1}{2} = \frac{5}{32}.$$

(b) Let events A and B_2 be defined as in the previous part. Using Bayes's Rule:

$$\mathbf{P}(B_2|A) = \frac{\mathbf{P}(A|B_2)\mathbf{P}(B_2)}{\mathbf{P}(A)}.$$

From the previous part, we said $\mathbf{P}(B_2) = \frac{1}{2}$, $\mathbf{P}(A|B_2) = \frac{1}{16}$, and $\mathbf{P}(A) = \frac{5}{32}$. Thus

$$\mathbf{P}(B_2|A) = \frac{\frac{1}{16} \cdot \frac{1}{2}}{\frac{5}{32}} = \frac{1}{5}.$$

As one would expect, given that Professor Right answers all the questions in a given lecture, it's more likely that she got only one question rather than two.

(c) We start by finding the PMFs for X and Y. The PMF $p_X(x)$ is given from the problem statement:

$$p_X(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in \{1,2\}; \\ 0, & \text{otherwise.} \end{cases}$$

The PMF for Y can be found by conditioning on X for each value that Y can take on. Because Professor Right can be asked at most two questions in any lecture, the range of Y is from 0 to 2. Looking at each possible value of Y, we find

$$p_Y(0) = \mathbf{P}(Y=0|X=0)\mathbf{P}(X=1) + \mathbf{P}(Y=0|X=2)\mathbf{P}(X=2) = \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{16} \cdot \frac{1}{2} = \frac{5}{32},$$

$$p_Y(1) = \mathbf{P}(Y=1|X=1)\mathbf{P}(X=1) + \mathbf{P}(Y=1|X=2)\mathbf{P}(X=2) = \frac{3}{4} \cdot \frac{1}{2} + 2 \cdot \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{9}{16},$$

$$p_Y(2) = \mathbf{P}(Y=2|X=1)\mathbf{P}(X=1) + \mathbf{P}(Y=2|X=2)\mathbf{P}(X=2) = 0 \cdot \frac{1}{2} + (\frac{3}{4})^2 \cdot \frac{1}{2} = \frac{9}{32}.$$

Note that when calculating $\mathbf{P}(Y=1|X=2)$, we got $2 \cdot \frac{3}{4} \cdot \frac{1}{4}$ because there are two ways for Professor Right to answer one question right when she's asked two questions: either

she answers the first question correctly or she answers the second question correctly. Thus, overall

$$
p_Y(y) = \begin{cases}
5/32, & \text{if } y = 0; \\
9/16, & \text{if } y = 1; \\
9/32, & \text{if } y = 2; \\
0, & \text{otherwise.}
\end{cases}
$$

Now the mean and variance can be calculated explicitly from the PMFs:
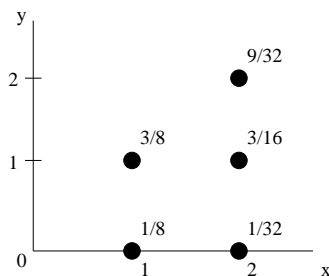
$$
\mathbf{E}[X] = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{3}{2},
$$

$$
\text{var}(X) = \left(1 - \frac{3}{2}\right)^2 \frac{1}{2} + \left(2 - \frac{3}{2}\right)^2 \frac{1}{2} = \frac{1}{4},
$$

$$
\mathbf{E}[Y] = 0 \cdot \frac{5}{32} + 1 \cdot \frac{9}{16} + 2 \cdot \frac{9}{32} = \frac{9}{8},
$$

$$
\text{var}(Y) = \left(0 - \frac{9}{8}\right)^2 \frac{5}{32} + \left(1 - \frac{9}{8}\right)^2 \frac{9}{16} + \left(2 - \frac{9}{8}\right)^2 \frac{9}{32} = \frac{27}{64}.
$$

(d) The joint PMF $p_{X,Y}(x,y)$ is plotted below. There are only five possible (x,y) pairs. For each point, $p_{X,Y}(x,y)$ was calculated by $p_{X,Y}(x,y) = p_X(x)p_{Y|X}(y|x)$.

(e) By linearity of expectations,

$$
\mathbf{E}[Z] = \mathbf{E}[X + 2Y] = \mathbf{E}[X] + 2\mathbf{E}[Y] = \frac{3}{2} + 2 \cdot \frac{9}{8} = \frac{15}{4}.
$$

Calculating $\text{var}(Z)$ is a little bit more tricky because X and Y are not independent; therefore we *cannot* add the variance of X to the variance of $2Y$ to obtain the variance of Z. (X and Y are clearly not independent because if we are told, for example, that $X = 1$, then we know that Y cannot equal 2, although normally without any information about X, Y could equal 2.)

To calculate $\text{var}(Z)$, first calculate the PMF for Z from the joint PDF for X and Y. For each (x,y) pair, we assign a value of Z. Then for each value z of Z, we calculate $p_Z(z)$ by summing over the probabilities of all (x,y) pairs that map to z. Thus we get

$$
p_Z(z) = \begin{cases}
1/8, & \text{if } z = 1; \\
1/32, & \text{if } z = 2; \\
3/8, & \text{if } z = 3; \\
3/16, & \text{if } z = 4; \\
9/32, & \text{if } z = 6; \\
0, & \text{otherwise.}
\end{cases}
$$

In this example, each (x, y) mapped to exactly one value of Z, but this does not have to be the case in general. Now the variance can be calculated as:

$$\text{var}(Z) = \frac{1}{8}\left(1 - \frac{15}{4}\right)^2 + \frac{1}{32}\left(2 - \frac{15}{4}\right)^2 + \frac{3}{8}\left(3 - \frac{15}{4}\right)^2 + \frac{3}{16}\left(4 - \frac{15}{4}\right) + \frac{9}{32}\left(6 - \frac{15}{4}\right)^2 = \frac{43}{16}.$$

(f) For each lecture i, let Z_i be the random variable associated with the number of questions Professor Right gets asked plus two times the number she gets right. Also, for each lecture i, let D_i be the random variable $1000 + 40Z_i$. Let S be her semesterly salary. Because she teaches a total of 20 lectures, we have

$$S = \sum_{i=1}^{20} D_i = \sum_{i=1}^{20} 1000 + 40Z_i = 20000 + 40\sum_{i=1}^{20} Z_i.$$

By linearity of expectations,

$$\mathbf{E}[S] = 20000 + 40\mathbf{E}[\sum_{i=1}^{20} Z_i] = 20000 + 40(20)\mathbf{E}[Z_i] = 23000.$$

Since each of the D_i are independent, we have

$$\text{var}(S) = \sum_{i=1}^{20} \text{var}(D_i) = 20\text{var}(D_i) = 20\text{var}(1000 + 40Z_i) = 20(40^2\text{var}(Z_i)) = 36000.$$

(g) Let Y be the number of questions she will answer wrong in a randomly chosen lecture. We can find $\mathbf{E}[Y]$ by conditioning on whether the lecture is in math or in science. Let M be the event that the lecture is in math, and let S be the event that the lecture is in science. Then

$$\mathbf{E}[Y] = \mathbf{E}[Y|M]\mathbf{P}(M) + \mathbf{E}[Y|S]\mathbf{P}(S).$$

Since there are an equal number of math and science lectures and we are choosing randomly among them, $\mathbf{P}(M) = \mathbf{P}(S) = \frac{1}{2}$. Now we need to calculate $\mathbf{E}[Y|M]$ and $\mathbf{E}[Y|S]$ by finding the respective conditional PMFs first. The PMFs can be determined in an manner analogous to how we calculated the PMF for the number of correct answers in part (c).

$$p_{Y|S}(y) = \begin{cases} \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2}(\frac{3}{4})^2 = 21/32, & \text{if } y = 0; \\ \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 2 \cdot \frac{1}{4} \cdot \frac{3}{4} = 5/16, & \text{if } y = 1; \\ \frac{1}{2} \cdot 0 + \frac{1}{2}(\frac{1}{4})^2 = 1/32, & \text{if } y = 2; \\ 0, & \text{otherwise.} \end{cases}$$

$$p_{Y|M}(y) = \begin{cases} \frac{1}{2} \cdot \frac{9}{10} + \frac{1}{2}(\frac{9}{10})^2 = 171/200, & \text{if } y = 0; \\ \frac{1}{2} \cdot \frac{1}{10} + \frac{1}{2} \cdot 2 \cdot \frac{1}{10} \cdot \frac{9}{10} = 7/50, & \text{if } y = 1; \\ \frac{1}{2} \cdot 0 + \frac{1}{2}(\frac{1}{10})^2 = 1/200, & \text{if } y = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\mathbf{E}[Y|S] = 0 \cdot \frac{21}{32} + 1 \cdot \frac{5}{16} + 2 \cdot \frac{1}{32} = \frac{3}{8},$$

$$\mathbf{E}[Y|M] = 0 \cdot \frac{171}{200} + 1 \cdot \frac{7}{50} + 2 \cdot \frac{1}{200} = \frac{3}{20}.$$

This implies that

$$\mathbf{E}[Y] = \frac{3}{20} \cdot \frac{1}{2} + \frac{3}{8} \cdot \frac{1}{2} = \frac{21}{80}.$$

2. The key to the problem is the even symmetry with respect to x of $p_{X,Y}(x,y)$ combined with the odd symmetry with respect to x of xy^3.

   By definition,

   $$\mathbf{E}[XY^3] = \sum_{x=-5}^{5} \sum_{y=0}^{10} xy^3 p_{X,Y}(x,y).$$

   All the $x = 0$ terms make no contribution to sum above because $\left[xy^3 p_{X,Y}(x,y)\right]\big|_{x=0} = 0$. The term for any pair (x,y) with $x \neq 0$ can be paired with the term for $(-x,y)$. These terms cancel, so the summation above is zero. Thus, without making in more detailed computations we can conclude $\mathbf{E}[XY^3] = 0$.

3. (a) Let L_i be the event that Joe played the lottery on week i, and let W_i be the event that he won on week i. We are asked to find

   $$\mathbf{P}(L_i \mid W_i^c) = \frac{\mathbf{P}(W_i^c \mid L_i)\mathbf{P}(L_i)}{\mathbf{P}(W_i^c \mid L_i)\mathbf{P}(L_i) + \mathbf{P}(W_i^c \mid L_i^c)\mathbf{P}(L_i^c)} = \frac{(1-q)p}{(1-q)p + 1 \cdot (1-p)} = \frac{p - pq}{1 - pq}.$$

   (b) Conditioned on X, the random variable Y is binomial:

   $$p_{Y|X}(y \mid x) = \begin{cases} \binom{x}{y} q^y (1-q)^{(x-y)}, & 0 \leq y \leq x; \\ 0, & \text{otherwise.} \end{cases}$$

   (c) Realizing that X has a binomial PMF, we have

   $$\begin{aligned} p_{X,Y}(x,y) &= p_{Y|X}(y \mid x)p_X(x) \\ &= \begin{cases} \binom{x}{y} q^y (1-q)^{(x-y)} \binom{n}{x} p^x (1-p)^{(n-x)}, & 0 \leq y \leq x \leq n; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

   (d) Using the result from (c), we could compute

   $$p_Y(y) = \sum_{x=y}^{n} p_{X,Y}(x,y),$$

   but the algebra is messy. An easier method is to realize that Y is just the sum of n independent Bernoulli random variables, each having a probability pq of being 1. Therefore Y has a binomial PMF:

   $$p_Y(y) = \begin{cases} \binom{n}{y} (pq)^y (1-pq)^{(n-y)}, & 0 \leq y \leq n; \\ 0, & \text{otherwise.} \end{cases}$$

(e)

$$p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

$$= \begin{cases} \dfrac{\begin{pmatrix} x \\ y \end{pmatrix} q^y(1-q)^{(x-y)} \begin{pmatrix} n \\ x \end{pmatrix} p^x(1-p)^{(n-x)}}{\begin{pmatrix} n \\ y \end{pmatrix} (pq)^y(1-pq)^{(n-y)}}, & 0 \le y \le x \le n; \\[2em] 0, & \text{otherwise.} \end{cases}$$

(f) Given $Y = y$, we know that Joe played y weeks with certainty. For each of the remaining $n - y$ weeks that Joe did not win there are $x - y$ weeks where he played. Each of these events occured with probability $\mathbf{P}(L_i \mid W_i^c)$ (the answer from part (a)). Using this logic we see that that X conditioned on Y is binomial:

$$p_{X|Y}(x \mid y) = \begin{cases} \begin{pmatrix} n-y \\ x-y \end{pmatrix} \left(\frac{p-pq}{1-pq}\right)^{x-y} \left(1 - \frac{p-pq}{1-pq}\right)^{n-x}, & 0 \le y \le x \le n; \\[1em] 0, & \text{otherwise.} \end{cases}$$

After some algebraic manipulation, the answer to (e) can be shown to be equal to the one above.

4. (a) V and W cannot be independent. Knowledge of one random variable gives information about the other. For instance, if $V = 12$ we know that $W = 0$.

(b) We begin by drawing the joint PMF of X and Y.
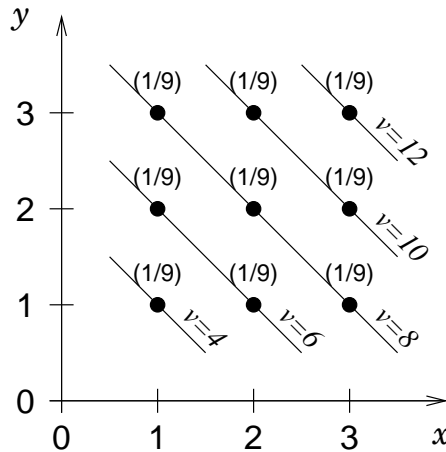
X and Y are uniformly distributed so each of the nine grid points has probability $1/9$. The lines on the graph represent areas of the sample space in which V is constant. This constant value of V is indicated on each line. The PMF of V is calculated by adding the probability associated with each grid point on the appropriate line.
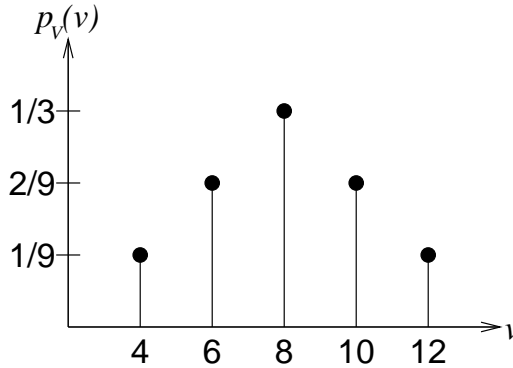
By symmetry (or direct calculation), $\boxed{E[V] = 8}$. The variance is:

$$\mathrm{var}(V) = (4-8)^2 \cdot \frac{1}{9} + (6-8)^2 \cdot \frac{2}{9} + 0 + (10-8)^2 \cdot \frac{2}{9} + (12-8)^2 \cdot \frac{1}{9} = \boxed{\frac{16}{3}}.$$

Alternatively, note that V is twice the sum of two independent random variables, $V = 2(X+Y)$, and hence

$$\mathrm{var}(V) = \mathrm{var}\big(2(X+Y)\big) = 2^2\mathrm{var}(X+Y) = 4\big(\mathrm{var}(X)+\mathrm{var}(Y)\big) = 4{\cdot}2\,\mathrm{var}(X) = 8\,\mathrm{var}(X).$$

(Note the use of independence in the third equality; in the fourth one we use the fact that X and Y are identically distributed, therefore they have the same variance). Now, by the distribution of X, we can easily calculate that
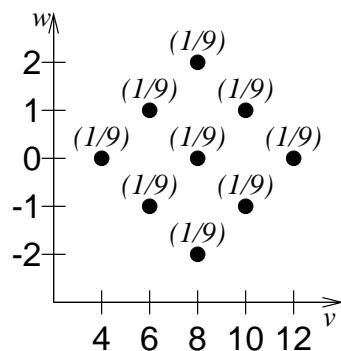
$$\mathrm{var}(X) = \frac{1}{3}(1-2)^2 + \frac{1}{3}(2-2)^2 + \frac{1}{3}(3-2)^2 = \frac{2}{3},$$

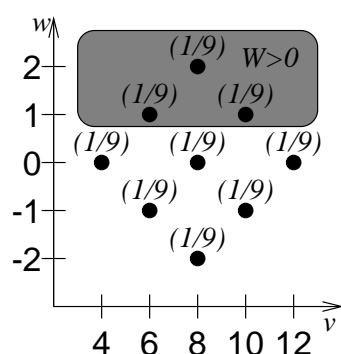so that in total $\mathrm{var}(V) = \frac{16}{3}$, as before.

(c) We start by adding lines corresponding to constant values of W to our first graph in part (b):
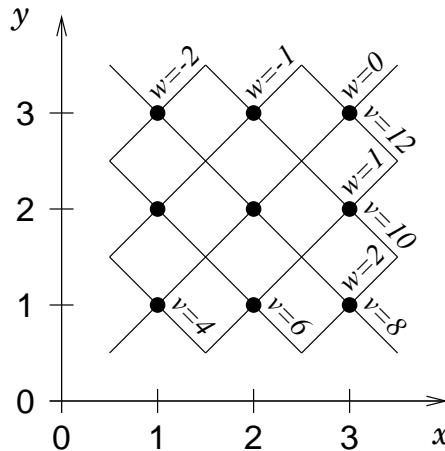
Again, each grid point has probability $1/9$. Using the above graph, we get $p_{V,W}(v,w)$.
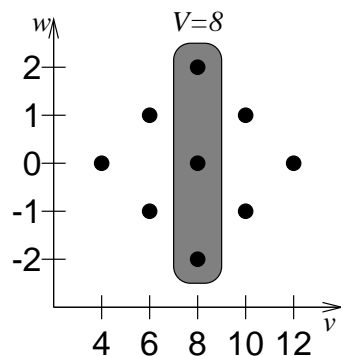
(d) The event $W > 0$ is shaded below:

By symmetry (or an easy calculation), $\boxed{\mathbf{E}[V \mid W > 0] = 8}$.

(e) The event $\{V = 8\}$ is shaded below:

When $V = 8$, W can take on values in the set $\{-2, 0, 2\}$ with equal probability. By symmetry (or an easy calculation), $\mathbf{E}[W \mid V = 8] = 0$. The variance is:

$$\text{var}(W \mid V = 8) = (-2 - 0)^2 \cdot \frac{1}{3} + (0 - 0)^2 \cdot \frac{1}{3} + (2 - 0)^2 \cdot \frac{1}{3} = \boxed{\frac{8}{3}} .$$

(f) Please refer to the first graph in part (b). When $V = 4$, $X = 1$ with probability 1. When $V = 6$, X can take on values in the set $\{1, 2\}$ with equal probability. Continuing this reasoning for the other values of V, we get the following conditional PMFs, which we plot on one set of axes.
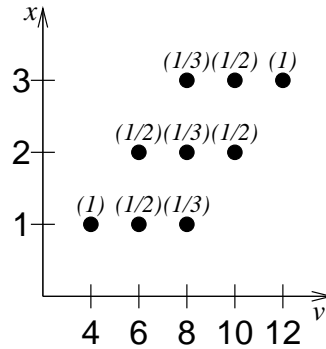
Note that each column of the graph is a separate conditional PMF and that the probability of each column sums to 1. This part of the problem illustrates an important point. $p_{X|V}(x \mid v)$ is actually not a single PMF but a family of PMFs.
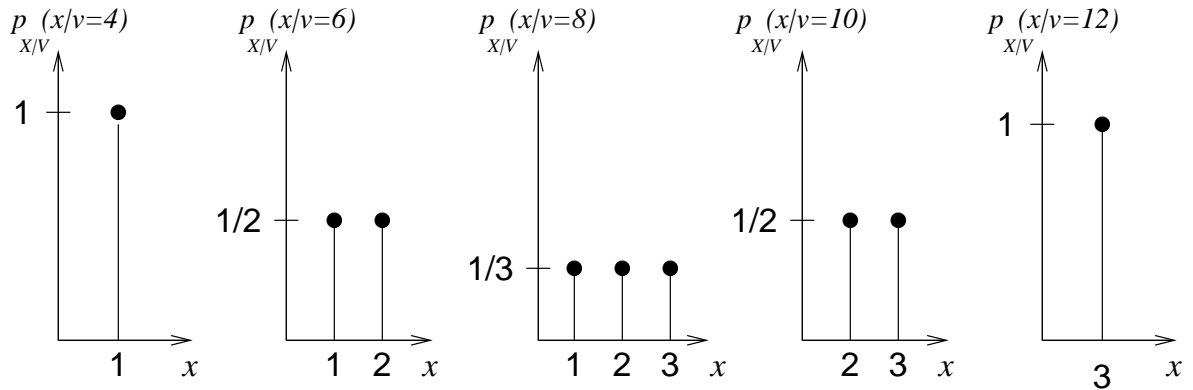
5. This probem asks for basic use of the probability density function.

   (a) The probability that you wait more than 15 minutes is:

   $$\int_{15}^{\infty} \frac{1}{15} e^{-\frac{x}{15}} \, dx = -e^{-\frac{x}{15}} \Big|_{15}^{\infty} = e^{-1}.$$

   (b) The probability that you wait between 15 and thirty minutes is:

   $$\int_{15}^{30} \frac{1}{15} e^{-\frac{x}{15}} \, dx = -e^{-\frac{x}{15}} \Big|_{15}^{30} = e^{-1} - e^{-2}.$$

G1†. (a) Clark must drive 12 block lengths to work, and each path is uniquely defined by saying which 7 of those 12 block lengths is traveled East. For instance, the path indicated by the heavy line is (E,E,E,E,E,N,N,N,N,N,E,E). There are a total of $\binom{12}{7}$ or 792 paths. The probability of picking any one path is $\frac{1}{792}$.

   (b) Out of 792 paths, the ones of interest travel from point A to C and then from point C to point B. Paths from A to C: $\binom{7}{4} = 35$. Paths from C to B: $\binom{5}{3} = 10$. Paths from A to C to B: 350. Thus, the probability of passing through C is $\frac{350}{792}$.

   (c) If Clark was seen at point D, he must have reached the intersection of 3rd St. and 3rd Ave. The reasoning from this point is similar to that above. Paths from intersection to B: $\binom{6}{4} = 15$. Paths from intersection to C to B: $\binom{5}{3} = 10$. Thus the conditional probability of passing through C after being sighted at D is $\frac{2}{3}$.

(d) Paths from Main and Broadway to wth and hth: $\binom{w+h}{w}$.
Paths from Main and Broadway to xth and yth: $\binom{x+y}{x}$.
Paths from x and y to wth and hth: $\binom{(w-x)+(h-y)}{(w-x)}$.

Thus the probability of passing by the phone booth is $\dfrac{\binom{x+y}{x}\binom{(w-x)+(h-y)}{(w-x)}}{\binom{w+h}{w}}$.

  i. If $w = 0$, then $x = 0$. The probability is $\dfrac{\binom{y}{0}\binom{(h-y)}{0}}{\binom{h}{0}} = 1$, which is reasonable since Clark must pass by the phone booth if there's only one route to work.

 ii. If $h = 0$, then $y = 0$. Similarly, the probability is 1.