Massachusetts Institute of Technology

Department of Electrical Engineering & Computer Science

6.041/6.431: Probabilistic Systems Analysis

(Spring 2006)

Problem Set 7: Solutions
Due: April 12, 2006

1. For both parts (a) and (b) we will make use of the formulas:

$$\begin{aligned} \mathbf{E}[X] &= \mathbf{E}[\mathbf{E}[X|Y]] \\ \text{var}(X) &= \mathbf{E}[\text{var}(X|Y)] + \text{var}(\mathbf{E}[X|Y]) \end{aligned}$$

Let X be the number of heads, and let Y be the result of the roll. Note that it can be easily verified that $E[Y] = 7/2$ and $Var(Y) = 35/12$.

(a)

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Y] = \mathbf{E}[Y/2] = \frac{E[Y]}{2} = \frac{7}{4}.$$

and similarly,

$$\text{var}(X) = \mathbf{E}[\text{var}(X|Y)] + \text{var}(\mathbf{E}[X|Y]) = \mathbf{E}[Y/4] + \text{var}(Y/2) = \frac{\mathbf{E}[Y]}{4} + \frac{var(Y)}{4} = \frac{77}{48}.$$

(b) For this part, let X_1 be the number of heads that correspond to the first die roll, and X_2 be the number of heads that correspond to the second die roll. Clearly $X = X_1 + X_2$ and X_1, X_2 are iid. Thus we have

$$\mathbf{E}[X] = \mathbf{E}[X_1 + X_2] = 2\mathbf{E}[X_1] = 2 \cdot \frac{7}{4} = \frac{7}{2}.$$

Similarly,

$$\text{var}(X) = \text{var}(X_1 + X_2) = 2\text{var}(X_1) = 2 \cdot \frac{77}{48} = \frac{77}{24}.$$

2. (a) Pat only needs to wait for Sam if Pat arrives before 9pm. If Pat arrives after 9pm, waiting time will simply be zero. Therefore, let T be the waiting time in hours,

$$\int_0^2 t f_X(x)dx = \int_0^1 (1-x)(\frac{1}{2})dx = \frac{1}{2}\left[x - \frac{1}{2}x^2\right]_0^1 = \frac{1}{4}$$

(b) Similar to part a), there are two cases. If Pat arrives before 9pm, the expected duration of the date will be 3 hours. Otherwise, the expected duration will be $\frac{3-X}{2}$, since the duration is uniformly distributed between 0 and $(3 - X)$ hours. Therefore,

$$\begin{aligned} \text{Expected duration} &= \int_0^1 (3)(\frac{1}{2})dx + \int_1^2 (\frac{3-x}{2})(\frac{1}{2})dx \\ &= \frac{3}{2} + \frac{1}{4}\left[3x - \frac{1}{2}x^2\right]_1^2 \\ &= \frac{15}{8} \end{aligned}$$

(c) The probability of having Pat late by more than 45 minutes on a date is $\frac{1}{8}$. Let U be the expected number of dates they will have before breaking up,
$U = Y_1 + Y_2$, where Y_1 and Y_2 are i.i.d. with geometric distribution ($p = \frac{1}{8}$). We know that $E[Y_1] = \frac{1}{p} = 8$. Therefore,

$$E[U] = E[Y_1] + E[Y_2] = 16.$$

3. Let D be the number of types of drinks the bartender makes, and let M be the number of people to enter the bar. Let $X_1, \ldots, X_N$ be the respective indicator variables of each drink. Thus if at least one person orders drink i, then $X_i = 1$ otherwise it equals 0. Note that $D = X_1 + \cdots + X_N$. Thus we have:

$$\begin{aligned} E[D] &= E[E[D|M=m]] \\ &= E[E[X_1 + \cdots + X_N|M=m]] \\ &= N \cdot E[E[X_i|M=m] \\ &= N \cdot E\Big[1 - \left(\frac{N-1}{N}\right)^m\Big] \\ &= N - N \cdot E\Big[\left(\frac{N-1}{N}\right)^m\Big] \\ \text{(letting } z = \tfrac{N-1}{N}\text{)} &= N - N \cdot E[Z^m] \\ &= N - N\sum_{k=0}^{\infty} z^k \cdot \frac{\lambda^k e^{-\lambda}}{k!} \\ &= N - Ne^{-\lambda} \cdot e^{\lambda z} \\ &= N - Ne^{-\frac{\lambda}{N}} \end{aligned}$$

4. (a) **From the definition of expectation:** $\mathbf{E}[\text{Yg(X)} \mid X] = \sum_y yg(X)p_{Y|X}(y|x)$
$= g(X)\sum_y yp_{Y|X}(y|x)$
$= g(X)\mathbf{E}[Y \mid X]$

(b) Show that

$$\mathbf{E}\left[\mathbf{E}[Y \mid X, Z] \mid X\right] = \mathbf{E}[Y \mid X] = \mathbf{E}\left[\mathbf{E}[Y \mid X] \mid X, Z\right]$$

Since

$$\mathbf{E}[Y \mid X, Z] = \sum_y yp(Y = y|X = x, Z = z)$$

From Law of Total Expectation $\mathbf{E}[\mathbf{E}[Y \mid X, Z] \mid X] = \sum_z \sum_y yp(Y = y|X = x, Z = z)p(X = x, Z = z|X = x)$
$= \sum_z \sum_y y\frac{p(Y=y,X=x,Z=z)}{p(X=x,Z=z)} \cdot \frac{p(X=x,Z=z)}{p(X=x)}$
$= \sum_z \sum_y y\frac{p(Y=y,X=x,Z=z)}{p(X=x)}$
$= \sum_z \sum_y y\frac{p(Y=y,X=x,Z=z)}{p(X=x)}$
$= \sum_y y\frac{p(Y=y,X=x)}{p(X=x)}$
$= \mathbf{E}[Y \mid X]$ From the Pull Through Property in part a. Let

$$g(X) = \mathbf{E}[Y \mid X] \quad and \quad \mathbf{E}[1 \mid Z] = 1$$

So, $\text{g(X)}\mathbf{E}[1 \mid Z] = \mathbf{E}[g(x) \mid X, Z]$
$= \mathbf{E}[\mathbf{E}[Y \mid X] \mid X, Z]$ The Pull-Through and Tower Properties are not limited to discrete random variables. These properties are also true in the continuous case. We can prove this by using the same approach we used for the discrete case.

5. (a) Since $E[X] = 0$, We have $E[E[X|Y]] = E[X] = 0$. Hence

$$\text{cov}(X, E[X \mid Y]) = E[XE[X \mid Y]] = E[E[XE[X \mid Y] \mid Y]] = E[(E[X \mid Y])^2] \geq 0.$$

(b) We can actually prove a stronger statement than what is asked for in the problem, namely that both pairs of random variables have the same covariance (from which it immediately follows that their correlation coefficients have the same sign. We have

$$\text{cov}(Y, E[X \mid Y]) = E[YE[X \mid Y]] = E[E[XY \mid Y]] = E[XY] = \text{cov}(X, Y).$$

6. (a) The transform $M_J(s)$ given is a transform of a binomial random variable with parameters $n = 10$ and $p = \frac{2}{3}$. Thus the PMF for J is:

$$p_J(j) = \binom{n}{j} (\frac{1}{3})^{n-j} (\frac{2}{3})^j \quad \text{for } j = 0, 1, 2, ...10$$

(b) Again by inspection, K is a geometric random variable shifted to the right by 3 with parameter $p = \frac{1}{5}$. This is because we can rewrite $M_K(s) = e^{3s} \frac{\frac{1}{5}e^s}{1-\frac{4}{5}e^s}$. Thus,

$$p_K(k) = (\frac{4}{5})^{k-4} \frac{1}{5} \quad \text{for } k = 4, 5, 6, \ ...$$

$$\mathbf{E}[K] = 3 + \frac{1}{p} = 3 + 5 = 8$$

$$\text{Var}(K) = \frac{1-p}{p^2} = \frac{\frac{4}{5}}{\frac{1}{25}} = 20$$

(c) Note that $L = K_1 + K_2 + ...K_J$, thus L is a random sum of random variables. So, determining the transform of L is easier than determining the PMF for L.

$$M_L(s) = M_J(s) \mid_{e^s = M_K(s)} = (\frac{1}{3} + \frac{2}{3}(\frac{\frac{1}{5}e^{4s}}{1 - \frac{4}{5}e^s}))^{10}$$

The expectation of L is $\mathbf{E}[L] = \mathbf{E}[K]\mathbf{E}[J] = 8 * \frac{20}{3} = \frac{160}{3}$
The variance of L is

$$\text{Var}(L) = \text{Var}(K)\mathbf{E}[J] + \text{Var}(J)(\mathbf{E}[K])^2 = (20)(10 * \frac{2}{3}) + (10 * \frac{2}{3} * \frac{1}{3})(64) = \frac{2480}{9}$$

(d) $\mathbf{P}(\text{person donates}) = \frac{1}{4}$. Let M = total # of donors from all living groups, and define

$$X_i = \begin{cases} 1 & \text{if ith person donates} \\ 0 & \text{otherwise.} \end{cases}$$

The PMF for X is just

$$p_X(x) = \begin{cases} \frac{1}{4} & \text{if } x = 1 \\ \frac{3}{4} & \text{if } x = 0. \end{cases}$$

Then,

$$M = X_1 + X_2 + ...X_L.$$

Therefore the transform of M is:

$$M_M(s) = M_L(s) \mid_{e^s = M_X(s)}$$

The transform of X is (by inspection)

$$M_X(s) = (\frac{3}{4} + \frac{1}{4}e^s)$$

Therefore,

$$M_M(s) = (\frac{1}{3} + \frac{2}{3}(\frac{\frac{1}{5}(\frac{3}{4} + \frac{1}{4}e^s)^4}{1 - \frac{4}{5}(\frac{3}{4} + \frac{1}{4}e^s)}))^{10}$$

To obtain $\mathbf{P}(M = 0)$, we simply evaluate the transform of M at $e^s = 0$.

$$p_M(0) = M_M(s) \mid_{e^s=0} = (\frac{1}{3} + \frac{2}{3}(\frac{\frac{1}{5}(\frac{3}{4})^4}{1 - \frac{4}{5}(\frac{3}{4})}))^{10}.$$

The expectation of M is $\mathbf{E}[M] = \mathbf{E}[X]\mathbf{E}[L] = \frac{40}{3}$
The variance of M is

$$\text{Var}(M) = \text{Var}(X)\mathbf{E}[L] + \text{Var}(L)(\mathbf{E}[X])^2 = 27.22$$

7. (a) Let the random variable T represent the number of widgets in 1 crate and let K_i represent the number of widgets in the ith carton.

$$T = K_1 + K_2 + ... + K_N$$

The transform of T is obtained by substituting the transform of N for every value of e^s in the transform of K.

$$M_T(s) = M_N(s) \mid_{e^s = M_K(s)}$$

$$M_T(s) = \frac{(1-p)e^{\mu(e^s-1)}}{1 - pe^{\mu(e^s-1)}}.$$

Since T is a non-negative discrete random variable,

$$\begin{aligned} P(T=1) &= \frac{d}{de^s} M_T(s) \mid_{e^s=0} \\ &= \frac{\mu(1-p)e^{-\mu}}{(1-pe^{-\mu})} + \frac{\mu p(1-p)e^{-2\mu}}{(1-pe^{-\mu})^2}. \end{aligned}$$

Since T is a non-negative discrete random variable, we can solve this problem using a different method.

$$\begin{aligned} M_T(s) &= p_T(0) + p_T(1)e^s + p_T(2)e^{2s} + p_T(3)e^{3s} + \ldots \\ M_T(s) - p_T(0) &= p_T(1)e^s + p_T(2)e^{2s} + p_T(3)e^{3s} + \ldots \\ \frac{M_T(s) - p_T(0)}{e^s} &= p_T(1) + p_T(2)e^s + p_T(3)e^{2s} + \ldots \\ p_T(1) &= \lim_{s \to -\infty} \frac{M_T(s) - p_T(0)}{e^s}. \end{aligned}$$

We can find $p_T(0)$ by taking the limit of the transform as $s \to -\infty$.

$$p_T(0) = \lim_{s\to-\infty} M_T(s) = \frac{(1-p)e^{-u}}{1-pe^{-u}}.$$

Substituting $p_T(0)$ and $M_T(s)$ to find $p_T(1)$ we get,

$$p_T(1) = \lim_{s\to-\infty} \frac{(1-p)e^{-\mu}\{e^{\mu e^s}(1-pe^{-\mu}) - (1-pe^{\mu(e^s-1)})\}}{e^s(1-pe^{\mu(e^s-1)})(1-pe^{-\mu})}$$

If we take the limit now the numerator and denominator both evaluate to 0. Therefore, we need to take the derivative of the numerator and denominator before we evaluate the limit.

$$\begin{aligned}
p_T(1) &= \lim_{s\to-\infty} \frac{(1-p)e^{-\mu}}{(1-pe^{-\mu})}\left[\frac{(1-pe^{-\mu})\mu e^{\mu e^s} + \mu pe^{\mu(e^s-1)}}{(1-pe^{\mu(e^s-1)}) - e^s(\mu pe^{\mu(e^s-1)})}\right] \\
p_T(1) &= \frac{(1-p)e^{-\mu}}{(1-pe^{-\mu})}\left[\frac{(1-pe^{-\mu})\mu + \mu pe^{-\mu}}{(1-pe^{-\mu})}\right] \\
p_T(1) &= \frac{\mu(1-p)e^{-\mu}}{(1-pe^{-\mu})} + \frac{\mu p(1-p)e^{-2\mu}}{(1-pe^{-\mu})^2}.
\end{aligned}$$

(b) Since K and N are independent,

$$\mathbf{E}[T] = \mathbf{E}[K]\mathbf{E}[N] = \frac{\mu}{1-p},$$

and

$$\begin{aligned}
\text{var}(T) &= \text{var}(K)\mathbf{E}[N] + (E[K])^2\text{var}(N) \\
&= \frac{\mu}{1-p} + \frac{\mu^2 p}{(1-p)^2}.
\end{aligned}$$

(c) Let W be the total weight of the widgets in a random crate.

$$W = X_1 + X_2 + ... + X_T$$

The transform of W is

$$M_W(s) = M_T(s) \mid_{e^s = M_X(s)}$$

$$M_W(s) = \frac{(1-p)e^{\mu(\frac{\lambda}{\lambda-s}-1)}}{1-pe^{\mu(\frac{\lambda}{\lambda-s}-1)}}.$$

(d) Since X and T are independent,

$$\mathbf{E}[W] = \mathbf{E}[X]\mathbf{E}[T] = \frac{\mu}{(1-p)\lambda},$$

and

$$\text{var}(w) = \text{var}(X)\mathbf{E}[T] + (E[X])^2\text{var}(T)$$

$$\frac{1}{\lambda^2}\left(\frac{2\mu}{(1-p)} + \frac{\mu^2 p}{(1-p)^2}\right)$$

8. (a) $\mathbf{P}(|X_1| \leq \delta) \approx \alpha\delta$.
$\mathbf{P}(-\delta \leq X_1 \leq \delta) = \int_{-\delta}^{\delta} f_{X_1}(x)dx_1 = 2\delta \cdot f_{X_1}(0) = \delta \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}}$.
$\alpha = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}}$.

(b) $\mathbf{E}[X_1 N] = \mathbf{E}[X_1]\mathbf{E}[N] = \frac{3}{2} \cdot 2 = 3$.

(c) $\text{var}(X_1 N) = \mathbf{E}[X_1^2 N^2] - (\mathbf{E}[X_1 N])^2 = (4+4)3 - 3^2 = 15$.

(d)

$$\begin{aligned} \mathbf{E}[X_1 + \cdots + X_N] &= \mathbf{E}[X_1 + \cdots + X_N \mid N \geq 2]\mathbf{P}(N \geq 2) + \\ &\quad \mathbf{E}[X_1 + \cdots + X_N \mid N < 2]\mathbf{P}(N < 2). \\ 3 &= \mathbf{E}[X_1 + \cdots + X_N \mid N \geq 2](1-p) + \mathbf{E}[X_1](p). \\ \mathbf{E}[X_1 + \cdots + X_N \mid N \geq 2] &= 3(3 - 2(2/3)) = 5. \end{aligned}$$

(e) Let $Z = N + X_1 + \cdots + X_N$. Note that N and $X_1 + \cdots + X_N$ are NOT independent.

$M_Z(s) = \mathbf{E}[\mathbf{E}[e^{s(N+X_1+\cdots+X_N)}|N]] = \mathbf{E}[\mathbf{E}[e^{sN} \cdot e^{s(X_1+\cdots+X_N)}|N]] = \mathbf{E}[e^{sN}(M_X(s))^N]$
$= \mathbf{E}[(e^s M_X(s))^N] = M_N(s)|_{e^s = e^s M_X(s)}$.

$M_N(s) = \frac{(2/3)e^s}{1-(1/3)e^s}$.

$M_X(s) = e^{2s^2+2s}$.

$M_Z(s) = \frac{(2/3)e^s e^{2s^2+2s}}{1-(1/3)e^s e^{2s^2+2s}} = \frac{2e^{2s^2+3s}}{3-e^{2s^2+3s}}$.

9. (a) Let $X = T_1 + T_2 + ... + T_N$ where N is a binomial with parameters $p = \frac{2}{3}$ and $n = 12$. $E[T_i] = \frac{1}{\lambda} = \frac{1}{3}$, thus, T_i is an exponential with rate $\lambda = 3$, so $f_{T_i}(t) = 3e^{-3t}$ with $t \geq 0$.

$E[X] = E[T_i]E[N] = \frac{1}{3} * 12 * \frac{2}{3} = \frac{8}{3}$

$var(X) = var(T_i)E[N] + (E[T_i])^2 var(N) = \frac{1}{9} * 12 * \frac{2}{3} + \frac{1}{9} * 12 * \frac{2}{3} * \frac{1}{3} = \frac{32}{27}$

(b) The fact that Iwana spends AT LEAST $\frac{1}{6}$ of an hour on each question shifts the transform in t by $\frac{1}{6}$, thus $f_{T_i}(t) = 3e^{-3(t-\frac{1}{6})}$ for $t \geq \frac{1}{6}$. We know that a shift by t in the pdf domain leads to a multiplation by e^{ts} in the transform domain. Therefore, the new $M_{T_i}(s) = \frac{3e^{\frac{s}{6}}}{3-s}$. Thus we have,

$$M_X(s) = M_N(s)|_{e^s = M_{T_i}(s)} = \left(\frac{1}{3} + \frac{2}{3}\frac{3e^{\frac{s}{6}}}{(s-3)}\right)^{12}$$

(c) By the law of iterated expectations, $E[N] = E[E[N|P]]$. We can compute $E[N|P]$ from the fact that N is a binomial with parameter P, where P is a random variable uniformly distributed between 0 and 1. Thus $E[N] = E[12P] = 12E[P] = 12 * \frac{1}{2} = 6$

We compute the $var(N)$ using the law of conditional variance: $var(N) = E[var(N|P)] + var(E[N|P]) = E[12P(1-P)] + var(12P)$

$= 12E[P(1-P)] + 144var(P)$
$= 12(1/2 - 1/3) + 12 = 14.$

(d)

$$M_N(s) = E[E[e^{sN}|P]] = E[M_P(s)] = E[1-P+Pe^s] = 1-E[P]+e^sE[P] = 1-\frac{1}{2}+\frac{1}{2}e^s = \frac{1}{2}+\frac{1}{2}e^s$$

10. Let A_t (respectively, B_t) be a Bernoulli random variable which is equal to 1 if and only if the tth toss resulted in 1 (respectively, 2). We have $\mathbf{E}[A_tB_t] = 0$ and $\mathbf{E}[A_tB_s] = \mathbf{E}[A_t]\mathbf{E}[B_s] = p_1p_2$ for $s \neq t$. We have

$$\mathbf{E}[X_1X_2] = \mathbf{E}[(A_1 + \cdots + A_n)(B_1 + \cdots + B_n)] = n\mathbf{E}[A_1(B_1 + \cdots + B_n)] = n(n-1)p_1p_2,$$

and

$$\text{cov}(X_1, X_2) = \mathbf{E}[X_1X_2] - \mathbf{E}[X_1]E[X_2] = n(n-1)p_1p_2 - np_1np_2 = -np_1p_2.$$

11. (a) Here it is easier to find the PDF of Y. Since Y is the sum of independent Gaussian random variables, Y is Gaussian with mean 2μ and variance $2\sigma_X^2 + \sigma_Z^2$.

(b) i. The transform of N is

$$M_N(s) = \frac{1}{11}(1 + e^s + e^{2s} + \cdots + e^{10s}) = \frac{1}{11}\sum_{k=0}^{10} e^{ks}$$

Since Y is the sum of

- a random sum of Gaussian random variables
- an independent Gaussian random variable,

$$\begin{aligned} M_Y(s) &= \left(M_N(s)|_{e^s=M_X(s)}\right)M_Z(s) = \left(\frac{1}{11}\sum_{k=0}^{10}(e^{s\mu+\frac{s^2\sigma_X^2}{2}})^k\right)e^{\frac{s^2\sigma_Z^2}{2}} \\ &= \left(\frac{1}{11}\sum_{k=0}^{10}e^{sk\mu+\frac{s^2k\sigma_X^2}{2}}\right)e^{\frac{s^2\sigma_Z^2}{2}} \\ &= \frac{1}{11}\sum_{k=0}^{10}e^{sk\mu+\frac{s^2(k\sigma_X^2+\sigma_Z^2)}{2}} \end{aligned}$$

In general, this is *not* the transform of a Gaussian random variable.

ii. One can differentiate the transform to get the moments, but it is easier to use the laws of iterated expectation and conditional variance:

$$\begin{aligned} \mathbf{E}Y &= \mathbf{E}X\mathbf{E}N + \mathbf{E}Z = 5\mu \\ \text{var}(Y) &= \mathbf{E}N\text{var}(X) + (\mathbf{E}X^2)\text{var}(N) + \text{var}(Z) = 5\sigma_X^2 + 10\mu^2 + \sigma_Z^2 \end{aligned}$$

iii. Now, the new transform for N is

$$M_N(s) = \frac{1}{9}(e^{2s} + \cdots + e^{10s}) = \frac{1}{9}\sum_{k=2}^{10} e^{ks}$$

Therefore,

$$\begin{aligned} M_Y(s) &= \left(M_N(s)|_{e^s=M_X(s)}\right)M_Z(s) = \left(\frac{1}{9}\sum_{k=2}^{10}(e^{s\mu+\frac{s^2\sigma_X^2}{2}})^k\right)e^{\frac{s^2\sigma_Z^2}{2}} \\ &= \left(\frac{1}{9}\sum_{k=2}^{10}e^{sk\mu+\frac{s^2k\sigma_X^2}{2}}\right)e^{\frac{s^2\sigma_Z^2}{2}} \\ &= \frac{1}{9}\sum_{k=2}^{10}e^{sk\mu+\frac{s^2(k\sigma_X^2+\sigma_Z^2)}{2}} \end{aligned}$$

(c) Given Y, the linear least-squared estimator of X_k is given by

$$\begin{aligned} \hat{X}_k &= \mathbf{E}X_k + \frac{\text{cov}(X_k,Y)}{\text{var}(Y)}(Y-\mathbf{E}Y) \\ &= \mu + \frac{\text{cov}(X_k,Y)}{\text{var}(Y)}(Y-\mathbf{E}Y). \end{aligned}$$

To determine the mean and variance of Y we first determine those of N:

$$\begin{aligned} \mathbf{E}N &= \left(\frac{1}{4}10+\frac{3}{4}5\right) \\ &= \frac{25}{4} \\ \text{var}(N) &= \mathbf{E}\text{var}(N|timeofday) + \text{var}(\mathbf{E}N|timeofday) \\ &= 10+\frac{75}{16} = \frac{235}{16} \end{aligned}$$

Now

$$\begin{aligned} \mathbf{E}Y &= \mathbf{E}\mathbf{E}Y|N = \mathbf{E}N\mathbf{E}X + \mathbf{E}Z \\ &= \mathbf{E}N\mathbf{E}X = \frac{25}{4}\mu \\ \text{var}(Y) &= \mathbf{E}N\text{var}(X) + (\mathbf{E}X^2)\text{var}(N) + \text{var}(Z) \\ &= \frac{25}{4}\sigma_X^2 + \frac{235}{16}\mu^2 + \sigma_Z^2 \\ \text{cov}(X_k,Y) &= \mathbf{E}(X_k-\mu)(Y-25\mu/4) \\ &= \mathbf{E}\mathbf{E}(X_k-\mu)(Y-25\mu/4)|N \end{aligned}$$

Since

$$\mathbf{E}(X_k-\mu)(Y-25\mu/4)|N = \begin{cases} \sigma_X^2 & \text{if } N\geq k, \\ 0 & \text{otherwise} \end{cases}$$

then

$$\begin{aligned} \text{cov}(X_k,Y) &= \sigma_X^2 P(N\geq k) \\ &= \sigma_X^2\begin{cases} 0.25*\sum_k \frac{10^k e^{-10}}{k!} + 0.75\frac{11-k}{11} & \text{if } k\leq 10 \\ 0.25*\sum_k \frac{10^k e^{-10}}{k!} & \text{if } k>10 \end{cases} \end{aligned}$$