Massachusetts Institute of Technology
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Final Exam | Fall 2009)

Problem 2. (20 points)
A pair of jointly continuous random variables, X and Y, have a joint probability density function given by

$$f_{X,Y}(x,y) = \begin{cases} c, & \text{in the shaded region of Fig. 1} \\ 0, & \text{elsewhere.} \end{cases}$$

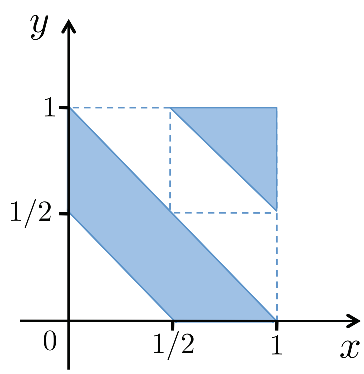


Figure 1: The shaded region is the domain in which $f_{X,Y}(x,y) = c$.

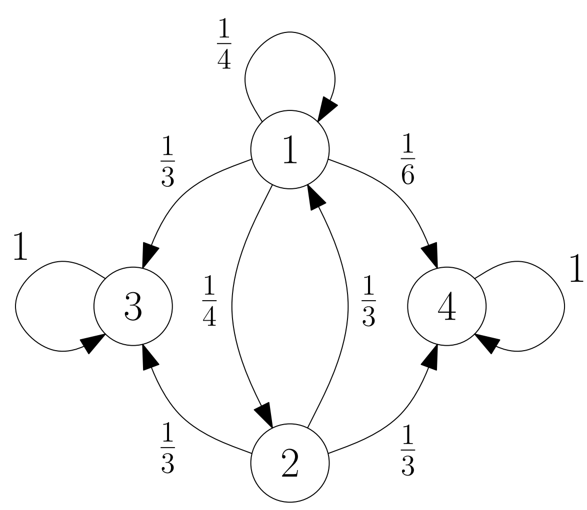
(a) (5 points) Find c.

(b) (5 points) Find the marginal PDFs of X and Y, i.e., $f_X(x)$ and $f_Y(y)$.

(c) (5 points) Find $\mathbf{E}[X \mid Y = 1/4]$ and $\mathrm{Var}[X \mid Y = 1/4]$, that is, the conditional mean and conditional variance of X given that $Y = 1/4$.

(d) (5 points) Find the conditional PDF for X given that $Y = 3/4$, i.e., $f_{X|Y}(x \mid 3/4)$.

Problem 3. (25 points)
Consider a Markov chain X_n whose one-step transition probabilities are shown in the figure.



(a) (5 points) What are the recurrent states?

(b) (5 points) Find $\mathbf{P}(X_2 = 4 \mid X_0 = 2)$.

(c) (5 points) Suppose that you are given the values of $r_{ij}(n) = \mathbf{P}(X_n = j \mid X_0 = i)$. Give a formula for $r_{11}(n+1)$ in terms of the $r_{ij}(n)$.

(d) (5 points) Find the steady-state probabilities $\pi_j = \lim_{n\to\infty} \mathbf{P}(X_n = j \mid X_0 = i)$, or explain why they do not exist.

(e) (5 points) What is the probability of eventually visiting state 4, given that the initial state is $X_0 = 1$?

Problem 4. (30 points)
Al, Bonnie, and Clyde run laps around a track, with the duration of each lap (in hours) being exponentially distributed with parameters $\lambda_A = 21$, $\lambda_B = 23$, and $\lambda_C = 24$, respectively. Assume that all lap durations are independent. At the completion of each lap, a runner drinks either one or two cups of water, with probabilities 1/3 and 2/3, respectively, independent of everything else, including how much water was consumed after previous laps. (The time spent drinking is negligible, assumed zero.)

(a) (5 points) Write down the PMF of the total number of completed laps over the first hour.

(b) (5 points) What is the expected number of cups of water to be consumed by the three runners, in total, over the first hour.

(c) (5 points) Al has amazing endurance and completed 72 laps. Find a good approximation for the probability that he drank at least 130 cups. (You do not have to use 1/2-corrections.)

(d) (5 points) What is the probability that Al finishes his first lap before any of the others?

(e) (5 points) Suppose that the runners have been running for a very long time when you arrive at the track. What is the distribution of the duration of Al's current lap? (This includes the duration of that lap both before and after the time of your arrival.)

(f) (5 points) Suppose that the runners have been running for 1/4 hours. What is the distribution of the time Al spends on his second lap, given that he is on his second lap?

Problem 5. (25 points)
A pulse of light has energy X that is a second-order Erlang random variable with parameter λ, i.e., its PDF is

$$f_X(x) = \begin{cases} \lambda^2 x e^{-\lambda x}, & \text{for } x \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

This pulse illuminates an ideal photon-counting detector whose output N is a Poisson-distributed random variable with mean x when $X = x$, i.e., its conditional PMF is

$$p_{N|X}(n \mid x) = \begin{cases} \dfrac{x^n e^{-x}}{n!}, & \text{for } n = 0, 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

(a) (5 points) Find $\mathbf{E}[N]$ and $\mathrm{Var}[N]$, the unconditional mean and variance of N

(b) (5 points) Find $p_N(n)$, the unconditional PMF of N.

(c) (5 points) Find $\hat{X}_{\text{lin}}(N)$, the linear least-squares estimator of X based on an observation of N.

(d) (5 points) Find $\hat{X}_{\text{MAP}}(N)$, the MAP estimator of X based on an observation of N.

(e) (5 points) Instead of the prior distribution in Eq. (1), we are now told that

$$\mathbf{P}(X=2) = 3^3/35, \qquad \mathbf{P}(X=3) = 2^3/35.$$

Given the observation $N = 3$, and in order to minimize the probability of error, which one of the two hypotheses $X = 2$ and $X = 3$ should be chosen?

Useful integral and facts:

$$\int_0^\infty y^k e^{-\alpha y}\,dy = \frac{k!}{\alpha^{k+1}}, \quad \text{for } \alpha > 0 \text{ and } k = 0, 1, 2, \ldots \text{ (recall that } 0!{=}1)$$

The second-order Erlang random variable satisfies:

$$\mathbf{E}[X] = 2/\lambda, \qquad \mathrm{Var}(X) = 2/\lambda^2.$$

Each question is repeated in the following pages. Please write your answer on the appropriate page.

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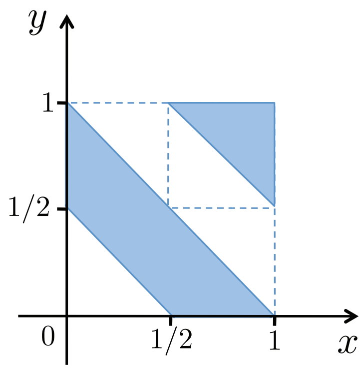

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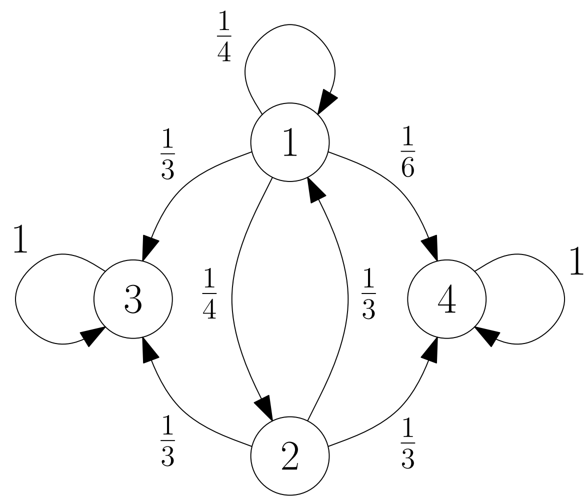

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