6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013
Transcript – Recitation: Uniform Probabilities on a Square

Hi. Today we're going to do another fun problem that involves rolling two dice. So if you guys happen to frequent casinos, this problem might be really useful for you. I'm just kidding.

But in all seriousness, this problem is a good problem, because it's going to remind us how and when to use the discrete uniform law. Don't worry, I'll review what that says. And it's also going to exercise your understanding of conditional probability.

So quick recap. The discrete uniform law says that when your sample space is discrete, and when the outcomes in your sample space are equally likely, then to compute the probability of any event A, you can simply count the number of outcomes in A and divide it by the total number of possible outcomes. OK, so coming back to our problem. The problem statement tells us that we roll two fair six-sided die. And it also tells us that each one of the 36 possible outcomes is assumed to be equally likely.

So you know alarm bell should be going off in your head. Our sample space is clearly discrete. And it says explicitly that all outcomes are equally likely. So clearly, we can use the discrete uniform law. And again, this is helpful because it reduces a problem of computing probabilities to a problem of counting.

OK, and before we go any further, I just want to review what this graph is plotting. You've seen it a few times, but just to clarify, on one axis, we're plotting the outcome of the first die roll, and on the second axis, we're plotting the outcome of the second die roll. So if you got a 4 on your first die, and you get a 1 on your second die, that corresponds to this point over 4 and up 1.

OK, so part a asks us to find the probability that doubles our rolls. So let's use some shorthand. We're going to let D be the event that doubles are rolled. And we want to compute the probability of D.

I argue before we can use the discrete uniform law. So if we apply that, we just get the number of outcomes that comprise the event "doubles rolled" divided by 36, because there are 36 possible outcomes, which you can see just by counting the dots in this graph. Six possible outcomes for the first die, six possible outcomes for the second die. That's how you-- 6 times 6 is 36.

So I've been assuming this entire time that you know what doubles are. For those of you who don't know, doubles is essentially when that number on the first die matches the number on the second die. So this outcome here 1-1 is part of the event "doubles rolled." Similarly, 2-2, 3-3, 4-4, 5-5, and 6-6-- these six points comprise the event "doubles rolled."

So we can go ahead and put 6 over 36, which is equal to 1/6. So we're done with part a. We haven't seen any conditioning yet. The conditioning comes in part b.

So in part b we're still interested in the event D, in the event that "doubles are rolled." But now we want to compute this probability conditioned on the event that the sum of the results is less than or equal to 4. So I'm going to use this shorthand sum less than or equal to 4 to denote the event that the role results in the sum of 4 or smaller.

So there's two ways we're going to go about solving part b. Let's just jump right into the first way. The first way is applying the definition of conditional probability. So hopefully you remember that this is just probability of D intersect sum less than or equal to 4, divided by probability of sum less than or equal to 4.

Now, sum less than or equal to 4 and D intersect sum less than or equal to 4 are just two events. And so we can apply the discrete uniform law to calculate both the numerator and the denominator. So let's start with the denominator first because it seems a little bit easier.

So sum less than or equal to 4, let's figure this out. Well, 1-1 gives us a sum of 2, that's less than or equal to 4. 2-1 gives us 3. 3-1 gives us 4. 4-1 gives us 5, so we don't want to include this or this, or this point.

And you can sort of convince yourself that the next point we want to include is this one. That corresponds to 2-2, which is 4, so it makes sense that these guys should form the boundary, because all dots sort of up and to the right will have a bigger sum.

3-1 gives us 4. And 1-2 gives us 3. So these six points-- 1, 2, 3, 4, 5, 6-- are the outcomes that comprise the event sum less than or equal to 4.

So we can go ahead and write in the denominator, 6 over 36, because we just counted the outcomes in sum less than or equal to, 4 and divided it by the number of outcomes in omega. Now, let's compute the numerator. D intersect sum less than or equal to 4. So we already found the blue check marks. Those correspond to sum less than or equal to 4.

Out of the points that have blue check marks, which one correspond to doubles? Well, they're actually already circled. It's just these two points. So we don't even need to circle those, so we get 2 over 36, using the discrete uniform law. And you see that these two 36s cancel each other. So you just get 2/6 or 1/3.

So that is one way of solving part b, but I want to take you, guys, through a different way, which I think is important, and that make sure you really understand what conditioning means. So another way that you can solve part b is to say, OK, we are now in the universe, we are in the conditional universe, where we know the sum of our results is 4 or smaller. And so that means our new sample space is really just this set of six points.

And one thing that it's worth noting is that conditioning never changes the relative frequencies or relative likelihoods of the different outcomes. So because all outcomes were equally likely in our original sample space omega, in the conditional worlds the outcomes are also equally likely. So using that argument, we could say that in our sort of blue conditional universe all of the outcomes are equally likely. And therefore, we can apply a conditional version of the discrete uniform law.

So namely, to compute the probability of some event in that conditional world. So the conditional probability that "doubles are rolled", we need only count the number of outcomes in that event and divide it by the total number of outcomes.

So in the conditional world, there's only two outcomes that comprise the event "doubles rolled." These are the only two circles in the blue region, right? So applying the conditional version number law, we have two. And then we need to divide by the size of omega. So our conditional universe, we've already said, has six possible dots. So we just divide by 6, and you see that we get the same answer of 1/3.

And so again, we used two different strategies. I happen to prefer the second one, because it's slightly faster and it makes you think about what does conditioning really mean. Conditioning means you're now restricting your attention to a conditional universe. And given that you're in this conditional universe where the sum was less than or equal to 4, what is then the probability that doubles also happened?

OK, hopefully you, guys, are following. Let's move on to part c. So part c asks for the probability that at least one die roll is a 6. So I'm going to use the letter S to denote this, the probability that at least one die roll is a 6.

So let's go back to our picture and we'll use a green marker. So hopefully you agree that anything in this column corresponds to at least one 6. So this point, this point, this point, this point, this point, and this point your first die landed on a 6, so at least one 6 is satisfied. Similarly, if your second die has a 6, then we're also OK.

So I claim we want to look at these 11 points. Let me just check that, yeah, 6 plus 5-- 11. So using the discrete uniform law again, we get 11 divided by 36.

OK, last problem, we're almost done. So again, we're interested in the event S again, so the event that at least one die roll is a 6. But now we want to compute the probability of that event in the conditional world where the two dice land on different numbers.

So I'm going to call this probability of S. Let's see, I'm running out of letters. Let's for lack of a better letter, my name is Katie, so we'll just use a K. We want to compute the probability of S given K. And instead of using the definition of conditional probability, like we did back in part b, we're going to use the faster route.

So essentially, we're going to find the number of outcomes in the conditional world. And then we're also going to compute the number of outcomes that comprise S in the conditional world. So let's take a look at this. We are conditioning on the event that the two dice land on different numbers.

So hopefully you agree with me that every single dot that is not on the diagonal, so every single dot that doesn't correspond to doubles, is a dot that we care about. So our conditional universe of that the two dice land on "different numbers", that corresponds to these dots. And it corresponds to these dots. I don't want to get this one. OK, that's good.

So let's see, how many outcomes do we have in our conditional world? And I'm sorry I don't know why I didn't include this. This is absolutely included. I'm just testing to see if you, guys, are paying attention.

So we counted before that there are six dots on the diagonal, and we know that there are 36 dots total. So the number of dots, or outcomes to use the proper word, in our conditional world is 36 minus 6, or 30. So we get a 30 on the denominator.

And now we're sort of using a conditional version of our discrete uniform law, again. And the reason why we can do this is, as I argued before, that conditioning doesn't change the relative frequency of the outcomes. So in this conditional world, all of the outcomes are still equally likely, hence we can apply this law again.

So now we need to count the number of outcomes that are in the orange conditional world, but that also satisfy at least one die roll is a 6. So you can see-- 1-- we just need to count the green circles that are also in the orange. So that's 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. So we get a 10, so our answer is 10 over 30, or 1/3.

So now we're done with this problem. As you see, hopefully, it wasn't too painful. And what are the important takeaways here for this problem?

Well, one is that whenever you have a discrete sample space, in which all of outcomes are equally likely, you should think about using the discrete uniform law, because this law lets you reduce the problem from computing probabilities to just counting outcomes within events. And the second takeaway is the way we thought about conditioning.

So we talked about one thing, which is that in your conditional world, when you condition, the relative likelihoods of the various outcomes don't change. So in our original universe, all of the outcomes were equally likely. So in our conditional universe, all of the outcomes are equally likely. And we saw it was much faster to apply a conditional version of the discrete uniform law. So that's it for today. And we'll do more problems next time.