Recitation 10 Solutions
(6.041/6.431 Spring 2010 Quiz 1 Solutions)

Question 1

1.1. Which **one** of the following statements is **true**?

(a) $\mathbf{P}(A \cap B)$ may be larger than $\mathbf{P}(A)$.

(b) The variance of X may be larger than the variance of $2X$.

(c) If $A^c \cap B^c = \emptyset$, then $\mathbf{P}(A \cup B) = 1$.

(d) If $A^c \cap B^c = \emptyset$, then $\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)$.

(e) If $\mathbf{P}(A) > 1/2$ and $\mathbf{P}(B) > 1/2$, then $\mathbf{P}(A \cup B) = 1$.

Answer: (c) is true because $A \cup B = (A^c \cap B^c)^c = \emptyset^c = \Omega$.

1.2. Which **one** of the following statements is **true**?

(a) If $\mathbf{E}[X] = 0$, then $\mathbf{P}(X > 0) = \mathbf{P}(X < 0)$.

(b) $\mathbf{P}(A) = \mathbf{P}(A \mid B) + \mathbf{P}(A \mid B^c)$

(c) $\mathbf{P}(B \mid A) + \mathbf{P}(B \mid A^c) = 1$

(d) $\mathbf{P}(B \mid A) + \mathbf{P}(B^c \mid A^c) = 1$

(e) $\mathbf{P}(B \mid A) + \mathbf{P}(B^c \mid A) = 1$

Answer: (e) is true because B and B^c partition Ω.

Question 2

Heather and Taylor play a game using independent tosses of an unfair coin. A head comes up on any toss with probability p, where $0 < p < 1$. The coin is tossed repeatedly until either the second time head comes up, in which case Heather wins; or the second time tail comes up, in which case Taylor wins. Note that a full game involves 2 or 3 tosses.

2.1. Consider a probabilistic model for the game in which the outcomes are the sequences of heads and tails in a full game. Provide a list of the outcomes and their probabilities of occurring.

Because of the independence of the coin tosses, the outcomes and their probabilities are as follows:

HH	p^2
HTH	$p^2(1-p)$
HTT	$p(1-p)^2$
THH	$p^2(1-p)$
THT	$p(1-p)^2$
TT	$(1-p)^2$

2.2. What is the probability that Heather wins the game?

The event of Heather winning is {HH, HTH, THH}. Adding the probabilities of the outcomes in this event gives $p^2 + p^2(1-p) + p^2(1-p) = p^2(3-2p)$.

2.3. What is the conditional probability that Heather wins the game given that head comes up on the first toss?

$$\begin{aligned}
\mathbf{P}(\{\textbf{Heather wins}\} \mid \{\textbf{first toss H}\}) &= \frac{\mathbf{P}(\{\textbf{Heather wins}\} \cap \{\textbf{first toss H}\})}{\mathbf{P}(\{\textbf{first toss H}\})} \\
&= \frac{\mathbf{P}(\{\textbf{HH, HTH}\})}{\mathbf{P}(\{\textbf{first toss H}\})} \\
&= \frac{p^2 + p^2(1-p)}{p} \;=\; p(2-p)
\end{aligned}$$

2.4. What is the conditional probability that head comes up on the first toss given that Heather wins the game?

$$\begin{aligned}
\mathbf{P}(\{\textbf{first toss H}\} \mid \{\textbf{Heather wins}\}) &= \frac{\mathbf{P}(\{\textbf{first toss H}\} \cap \{\textbf{Heather wins}\})}{\mathbf{P}(\{\textbf{Heather wins}\})} \\
&= \frac{\mathbf{P}(\{\textbf{HH, HTH}\})}{\mathbf{P}(\{\textbf{Heather wins}\})} \\
&= \frac{p^2 + p^2(1-p)}{p^2(3-2p)} \;=\; \frac{2-p}{3-2p}
\end{aligned}$$

Question 3

A casino game using a **fair** 4-sided die (with labels 1, 2, 3, and 4) is offered in which a **basic game** has 1 or 2 die rolls:

- If the first roll is a 1, 2, or 3, the player wins the amount of the die roll, in dollars, and the game is over.
- If the first roll is a 4, the player wins \$2 and the amount of a second ("bonus") die roll in dollars.

Let X be the payoff in dollars of the basic game.

3.1. Find the PMF of X, $p_X(x)$.

Define a probabilistic model in which the outcomes are the sequences of rolls in a full game. The outcomes, their probabilities, and the resulting values of X are as follows:

ω	$\mathbf{P}(\{\omega\})$	$X(\omega)$
(1)	1/4	1
(2)	1/4	2
(3)	1/4	3
$(4,1)$	1/16	3
$(4,2)$	1/16	4
$(4,3)$	1/16	5
$(4,4)$	1/16	6

By gathering the probabilities of the possible values for X, we obtain

$$p_X(x) = \begin{cases} 1/4, & \textbf{for } x = 1,2; \\ 5/16, & \textbf{for } x = 3; \\ 1/16, & \textbf{for } x = 4,5,6; \\ 0, & \textbf{otherwise}. \end{cases}$$

3.2. Find $\mathbf{E}[X]$.

It does not take too much arithmetic to compute $\mathbf{E}[X]$ using the PMF computed in the previous part. A more elegant solution is to use the total expectation theorem. Let A be the event that the first roll is a 4. Then

$$\mathbf{E}[X] = \underbrace{\mathbf{P}(A)}_{1/4}\,\underbrace{\mathbf{E}[X \mid A]}_{4.5} + \underbrace{\mathbf{P}(A^c)}_{3/4}\,\underbrace{\mathbf{E}[X \mid A^c]}_{2} = \frac{21}{8},$$

where $\mathbf{E}[X \mid A] = 4.5$ because the conditional distribution is uniform on $\{3,4,5,6\}$; and $\mathbf{E}[X \mid A^c] = 2$ because the conditional distribution is uniform on $\{1,2,3\}$.

3.3. Find the conditional PMF of the result of the first die roll given that $X = 3$. (Use a reasonable notation that you define explicitly.)

Let Z be the result of the first die roll, and let $B = \{X = 3\}$. By definition of conditioning,

$$p_{Z|B}(z) = \frac{\mathbf{P}(\{Z = z\} \cap B)}{\mathbf{P}(B)}.$$

By using values tabulated above,

$$p_{Z|B}(z) = \begin{cases} 4/5, & \textbf{for } z = 3; \\ 1/5, & \textbf{for } z = 4; \\ 0, & \textbf{otherwise}. \end{cases}$$

3.4. Now consider an **extended game** that can have any number of bonus rolls. Specifically:

- Any roll of a 1, 2, or 3 results in the player winning the amount of the die roll, in dollars, and the termination of the game.
- Any roll of a 4 results in the player winning \$2 and continuation of the game.

Let Y denote the payoff in dollars of the extended game. Find $\mathbf{E}[Y]$.

One could explicitly find the PMF of Y, but this is unnecessarily messy. Instead, let L be the payoff of the last roll and let W be the payoff of all of the earlier rolls. Then $Y = W + L$ by construction, and $\mathbf{E}[Y] = \mathbf{E}[W] + \mathbf{E}[L]$.

The last roll is uniformly distributed on $\{1,2,3\}$, so $\mathbf{E}[L] = 2$. The winnings on earlier rolls is $2(N-1)$ where N is the number of rolls in the game. Since termination of the game can be seen as "success" on a Bernoulli trial with success probability of $3/4$, N has the geometric distribution with parameter $3/4$. Thus,

$$\mathbf{E}[W] = \mathbf{E}[2(N-1)] = 2\mathbf{E}[N] - 2 = 2 \cdot \frac{4}{3} - 2 = \frac{2}{3}.$$

Combining the calculations,

$$\mathbf{E}[Y] = \mathbf{E}[W] + \mathbf{E}[L] = \frac{2}{3} + 2 = \frac{8}{3}.$$

(Many other methods of solution are possible.)