Problem 1. (80 points) In this problem:
(i) X is a (continuous) uniform random variable on $[0, 4]$.
(ii) Y is an exponential random variable, independent from X, with parameter $\lambda = 2$.

1. **(10 points)** Find the mean and variance of $X - 3Y$.

2. **(10 points)** Find the probability that $Y \geq X$.
   (Let c be the answer to this question.)

3. **(10 points)** Find the conditional joint PDF of X and Y, given that the event $Y \geq X$ has occurred.
   (You may express your answer in terms of the constant c from the previous part.)

4. **(10 points)** Find the PDF of $Z = X + Y$.

5. **(10 points)** Provide a fully labeled sketch of the conditional PDF of Z given that $Y = 3$.

6. **(10 points)** Find $\mathbf{E}[Z \mid Y = y]$ and $\mathbf{E}[Z \mid Y]$.

7. **(10 points)** Find the joint PDF $f_{Z,Y}$ of Z and Y.

8. **(10 points)** A random variable W is defined as follows. We toss a fair coin (independent of Y). If the result is "heads", we let $W = Y$; if it is tails, we let $W = 2 + Y$. Find the probability of "heads" given that $W = 3$.

Problem 2. (30 points) Let $X, X_1, X_2, \ldots$ be independent normal random variables with mean 0 and variance 9. Let N be a positive integer random variable with $\mathbf{E}[N] = 2$ and $\mathbf{E}[N^2] = 5$. We assume that the random variables $N, X, X_1, X_2, \ldots$ are independent. Let $S = \sum_{i=1}^{N} X_i$.

1. **(10 points)** If δ is a small positive number, we have $\mathbf{P}(1 \leq |X| \leq 1+\delta) \approx \alpha\delta$, for some constant α. Find the value of α.

2. **(10 points)** Find the variance of S.

3. **(5 points)** Are N and S uncorrelated? Justify your answer.

4. **(5 points)** Are N and S independent? Justify your answer.

Each question is repeated in the following pages. Please write your answer on the appropriate page.

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6.041SC Probabilistic Systems Analysis and Applied Probability
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