LECTURE 25
Outline

- **Reference:** Section 9.4

- Review of simple binary hypothesis tests
  - examples

- Testing composite hypotheses
  - is my coin fair?
  - is my die fair?
  - goodness of fit tests

Simple binary hypothesis testing

- **null hypothesis** H_0:
$$X \sim p_X(x; H_0) \quad \text{[or } f_X(x; H_0)]$$

- **alternative hypothesis** H_1:
$$X \sim p_X(x; H_1) \quad \text{[or } f_X(x; H_1)]$$

- Choose a **rejection region** R;
  reject H_0 iff data $\in R$

- Likelihood ratio test: reject H_0 if
$$\frac{p_X(x; H_1)}{p_X(x; H_0)} > \xi \quad \text{or} \quad \frac{f_X(x; H_1)}{f_X(x; H_0)} > \xi$$

- fix false rejection probability α
  (e.g., $\alpha = 0.05$)

- choose ξ so that $\mathbf{P}(\text{reject } H_0; H_0) = \alpha$

Example (test on normal mean)

- n data points, i.i.d.
  H_0: $X_i \sim N(0, 1)$
  H_1: $X_i \sim N(1, 1)$

- Likelihood ratio test; rejection region:
$$\frac{(1/\sqrt{2\pi})^n \exp\{-\sum_i (X_i - 1)^2 / 2\}}{(1/\sqrt{2\pi})^n \exp\{-\sum_i X_i^2 / 2\}} > \xi$$

- algebra: reject H_0 if: $\sum_i X_i > \xi'$

- Find ξ' such that
$$\mathbf{P}\Big(\sum_{i=1}^{n} X_i > \xi'; H_0 \Big) = \alpha$$

- use normal tables

Example (test on normal variance)

- n data points, i.i.d.
  H_0: $X_i \sim N(0, 1)$
  H_1: $X_i \sim N(0, 4)$

- Likelihood ratio test; rejection region:
$$\frac{(1/2\sqrt{2\pi})^n \exp\{-\sum_i X_i^2 / (2 \cdot 4)\}}{(1/\sqrt{2\pi})^n \exp\{-\sum_i X_i^2 / 2\}} > \xi$$

- algebra: reject H_0 if $\sum_i X_i^2 > \xi'$

- Find ξ' such that
$$\mathbf{P}\Big(\sum_{i=1}^{n} X_i^2 > \xi'; H_0 \Big) = \alpha$$

- the distribution of $\sum_i X_i^2$ is known
  (derived distribution problem)

- "chi-square" distribution;
  tables are available

Composite hypotheses

- Got $S = 472$ heads in $n = 1000$ tosses; is the coin fair?

  - $H_0 : p = 1/2$ versus $H_1 : p \neq 1/2$

- Pick a **"statistic"** (e.g., S)

- Pick shape of **rejection region** (e.g., $|S - n/2| > \xi$)

- Choose **significance level** (e.g., $\alpha = 0.05$)

- Pick **critical value** ξ so that:

$$\mathbf{P}(\text{reject } H_0; H_0) = \alpha$$

  Using the CLT:

$$\mathbf{P}(|S - 500| \leq 31; H_0) \approx 0.95; \quad \xi = 31$$

- In our example: $|S - 500| = 28 < \xi$
  H_0 **not rejected** (at the 5% level)

Is my die fair?

- Hypothesis H_0:
  $\mathbf{P}(X = i) = p_i = 1/6,\ i = 1, \ldots, 6$

- Observed occurrences of i: N_i

- Choose form of rejection region; chi-square test:

$$\text{reject } H_0 \text{ if } \quad T = \sum_i \frac{(N_i - np_i)^2}{np_i} > \xi$$

- Choose ξ so that:

$$\mathbf{P}(\text{reject } H_0; H_0) = 0.05$$

$$\mathbf{P}(T > \xi; H_0) = 0.05$$

- Need the distribution of T:
  (CLT + derived distribution problem)

  - for large n, T has approximately a chi-square distribution

  - available in tables

Do I have the correct pdf?

- Partition the range into bins
  - np_i: expected incidence of bin i (from the pdf)
  - N_i: observed incidence of bin i

  - Use chi-square test (as in die problem)

- Kolmogorov-Smirnov test:
  form **empirical CDF**, $\widehat{F}_X$, from data

(http://www.itl.nist.gov/div898/handbook/)
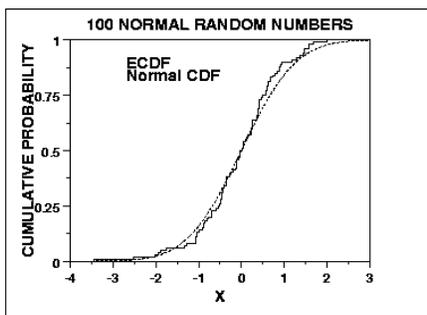
- $D_n = \max_x |F_X(x) - \widehat{F}_X(x)|$

- $\mathbf{P}(\sqrt{n} D_n \geq 1.36) \approx 0.05$

What else is there?

- Systematic methods for coming up with shape of rejection regions

- Methods to estimate an unknown PDF (e.g., form a histogram and "smooth" it out)

- Efficient and recursive signal processing

- Methods to select between less or more complex models
  - (e.g., identify relevant "explanatory variables" in regression models)

- Methods tailored to high-dimensional unknown parameter vectors and huge number of data points (data mining)

- etc. etc....

6.041SC Probabilistic Systems Analysis and Applied Probability
Fall 2013