

Solutions to In-Class Problems Week 10, Wed.

Problem 1. Find the coefficients of

(a) x^5 in $(1 + x)^{11}$

Solution.

$$\binom{11}{5} = 462$$

■

(b) x^8y^9 in $(3x + 2y)^{17}$

Solution.

$$\binom{17}{8} 3^8 2^9$$

■

(c) a^6b^6 in $(a^2 + b^3)^5$

Solution. $a^6b^6 = (a^2)^3(b^3)^2$, so the coefficient is

$$\binom{5}{3} = 10$$

■

Problem 2. According to the Multinomial theorem, $(w + x + y + z)^n$ can be expressed as a sum of terms of the form

$$\binom{n}{r_1, r_2, r_3, r_4} w^{r_1} x^{r_2} y^{r_3} z^{r_4}.$$

How many terms are there in the sum?

Solution. The sum is over all 4-tuples of nonnegative integers (r_1, r_2, r_3, r_4) such that

$$r_1 + r_2 + r_3 + r_4 = n.$$

We know this is the same as the number of binary words with n zeroes and 3 ones, namely

$$\binom{n+3}{3}.$$



Combinatorial proofs of identities

Recall the basic plan for a combinatorial proof of an identity $x = y$:

1. Define a set S .
2. Show that $|S| = x$ by counting one way.
3. Show that $|S| = y$ by counting another way.
4. Conclude that $x = y$.

Problem 3. You want to choose a team of m people from a pool of n people for your startup company, and from these m people you want to choose k to be the team managers. You took 6.042, so you know you can do this in

$$\binom{n}{m} \binom{m}{k}$$

ways. But your CFO, who went to Harvard Business School, comes up with the formula

$$\binom{n}{k} \binom{n-k}{m-k}.$$

Before doing the reasonable thing—dump on your CFO or Harvard Business School—you decide to check his answer against yours.

- (a) Start by giving an *algebraic proof* that your CFO's formula agrees with yours.

Solution.

$$\begin{aligned}
 \binom{n}{m} \binom{m}{k} &= \frac{n!}{m!(n-m)!} \frac{m!}{k!(m-k)!} \\
 &= \frac{n!}{(n-m)!k!(m-k)!} \\
 &= \frac{n!(n-k)!}{(n-m)!k!(m-k)!(n-k)!} \\
 &= \frac{n!}{k!(n-k)!} \frac{(n-k)!}{(n-m)!(m-k)!} \\
 &= \frac{n!}{k!(n-k)!} \frac{(n-k)!}{((n-k)-(m-k))!(m-k)!} \\
 &= \binom{n}{k} \binom{n-k}{m-k}.
 \end{aligned}$$

■

(b) Now give a *combinatorial argument* proving this same fact.

Solution. Instead of choosing first m from n and then k from m , you could alternately choose the k managers from the n people and then choose $m-k$ people to fill out the team from the remaining $n-k$ people. This gives you $\binom{n}{k} \binom{n-k}{m-k}$ ways of picking your team. Since you must have the same number of options regardless of the order in which you choose to pick team members and managers,

$$\binom{n}{m} \binom{m}{k} = \binom{n}{k} \binom{n-k}{m-k}.$$

■

Problem 4. Now give a combinatorial proof of the following, more interesting theorem:

$$n2^{n-1} = \sum_{k=1}^n k \binom{n}{k}$$

Hint: Let S be the set of all length- n sequences of 0's, 1's and a single *.

Solution. Let $P ::= \{0, \dots, n-1\} \times \{0, 1\}^{n-1}$. On the one hand, there is a bijection from P to S by mapping (k, x) to the word obtained by inserting a * just after the k th bit in the length- $n-1$ binary word, x . So

$$|S| = |P| = n2^{n-1} \tag{1}$$

by the Product Rule.

On the other hand, every sequence in S contains between 1 and n nonzero entries since the $*$, at least, is nonzero. The mapping from a sequence in S with exactly k nonzero entries to a pair consisting of the set of positions of the nonzero entries and the position of the $*$ among these entries is a bijection, and the number of such pairs is $\binom{n}{k}k$ by the Generalized Product Rule. Thus, by the Sum Rule:

$$|S| = \sum_{k=1}^n k \binom{n}{k}$$

Equating this expression and the expression (1) for $|S|$ proves the theorem. ■

Learning to count takes practice! The following problems offer some.

Problem 5. A pizza house is having a promotional sale. Their commercial reads:

We offer 9 different toppings for your pizza! Buy 3 large pizzas at the regular price, and you can get each one with as many different toppings as you wish, absolutely free. That's 22,369,621 different ways to choose your pizzas!

The ad writer was a former Harvard student who had evaluated the formula $(2^9)^3/3!$ on his calculator and gotten close to 22,369,621. Unfortunately, $(2^9)^3/3!$ is obviously not an integer, so clearly something is wrong. What mistaken reasoning might have led the ad writer to this formula? Explain how to fix the mistake and get a correct formula.

Solution. The number of ways to choose toppings for one pizza is the number of the possible subsets of the set of 9 toppings, namely, 2^9 . The ad writer presumably then used the Product Rule to conclude that there were $(2^9)^3$ sequences of three topping choices. Then he probably reasoned that each way of making three topping choices arises from $3!$ sequences, so the Division Rule would imply that the number of ways to choose three pizzas is $(2^9)^3/3!$.

It's true that every set of three *different* topping choices arises from $3!$ different length-3 sequences of choices. The mistake is that if some of the three choices are the same, then the set of three choices arises from *fewer* than $3!$ sequences. For example, if all three pizzas have the same toppings, there is only one sequence of topping choices for them.

One fix is to consider ways to choose toppings with 1, 2 and 3 different topping choices. There are $2^9(2^9 - 1)(2^9 - 2)/3!$ ways to choose a set of 3 different choices, $2^9(2^9 - 1)$ ways to choose one topping choice to be used on two pizzas and a second choice for the third pizza, and 2^9 ways to choose one topping for all three pizzas, giving

$$\frac{2^9(2^9 - 1)(2^9 - 2)}{3!} + 2^9(2^9 - 1) + 2^9 = 22,500,864.$$

ways to choose three pizzas.

Alternatively, we can observe that this is exactly the problem of selecting a dozen donuts of five possible different kinds – except now there are 3 donuts and 2^9 kinds. Hence, there is a bijection

to the number of $(2^9 + 2)$ -bit strings with exactly $2^9 - 1$ ones and 3 zeros:

$$\binom{2^9 + 2}{3} = 22,500,864.$$

■

Problem 6. (a) In how many different ways can Blockbuster arrange 64 copies of *13 conversations about one thing*, 96 copies of *L'Auberge Espagnole* and 1 copy of *Matrix Revolutions* on a shelf? What if they are to be arranged in 5 shelves?

Solution. For 1 shelf, this is the number of ways to arrange 64 *C*'s, 96 *A*'s, and 1 *M*. By the bookkeeper rule:

$$\frac{(64 + 96 + 1)!}{64! 96! 1!}$$

For 5 shelves, we can do the simple trick of introducing the dividers between the shelves as new objects. That is, we want the number of ways to arrange 64 *C*'s, 96 *A*'s, 1 *M*, and 4 *X*'s (dividers). By the bookkeeper theorem, again:

$$\frac{(64 + 96 + 1 + 4)!}{64! 96! 1! 4!}$$

■

(b) Set A has r elements and set B has n elements. How many functions are there from A to B ? How many of them are injective (one-to-one)? How many of them are bijective?

Solution. Say $A = \{a_1, \dots, a_r\}$ and $B = \{b_1, \dots, b_n\}$ and consider the mapping that sends every function $f : A \rightarrow B$ to the sequence $(f(a_1), \dots, f(a_r))$. This is a bijection between functions from A to B and r -long sequences of elements from B . By the product rule, the number of such sequences is

$$\underbrace{n \cdot n \cdots n}_{r \text{ times}} = n^r.$$

For injections, first note that (by the pigeonhole principle) there is no way to inject A into B if B has fewer elements than A . That is: if $r > n$, then the number of injections from A to B is 0. If $r \leq n$, though, the same mapping as previously becomes a bijection between injections from A to B and r -long sequences of *distinct* elements from B . By the generalized product rule, the number of such sequences is

$$n \cdot (n - 1) \cdot (n - 2) \cdots (n - r + 1) = \frac{n!}{(n - r)!},$$

that is, the number of r -permutations of n elements.

For bijections, we similarly note that in the case $r \neq n$ the number of bijections from A to B is 0. If $r = n$, then a function from A to B is a bijection iff it is an injection. So the number

of bijections equals the number of injections: $n!/(n-n)! = n!$, which is exactly the number of different permutations of n elements.

Notice how *functions*, *injections*, and *bijections* correspond respectively to *sequences*, *r-permutations*, and *permutations*. ■

(c) Find the number of 5-card hands in which every suit appears at most twice.

Solution. There are two cases. Either one suit appears twice or else two suits appear twice. The number of hands in which one suit appears twice is $\binom{13}{2} \cdot 13^3 \cdot 4$, since there are 4 ways to choose the doubly-represented suit, $\binom{13}{2}$ ways to choose two values from this suit, and 13^3 ways to choose values for cards in the three remaining suits. Similarly, the number of hands in which two suits appear twice is $\binom{13}{2}^2 \cdot 13 \cdot \binom{4}{2} \cdot 2$. Therefore, there are a total of

$$\binom{13}{2} \cdot 13^3 \cdot 4 + \binom{13}{2}^2 \cdot 13 \cdot \binom{4}{2} \cdot 2$$

such hands. ■

(d) How many paths are there from point $(0, 0)$ to $(50, 50)$ if every step increments one coordinate and leaves the other unchanged? if there are impassable boulders sitting at points $(10, 10)$ and $(20, 20)$? *Hint:* Count the number of paths going through $(10, 10)$, the number through $(20, 20)$, and use Inclusion-Exclusion.

Solution. We use inclusion-exclusion. The total number of paths is $\binom{100}{50}$, but we must subtract off the obstructed paths. There are $\binom{20}{10} \cdot \binom{80}{40}$ paths through the first boulder, since there are $\binom{20}{10}$ paths from the start to the boulder and $\binom{80}{40}$ paths from the boulder to the finish. Similarly, there are $\binom{40}{20} \cdot \binom{60}{30}$ paths through the second boulder. However, we must subtract off paths going through both boulders, and there are $\binom{20}{10} \cdot \binom{20}{10} \cdot \binom{60}{30}$ of those. Therefore, the total number of paths is:

$$\binom{100}{50} - \binom{20}{10} \cdot \binom{80}{40} - \binom{40}{20} \cdot \binom{60}{30} + \binom{20}{10} \cdot \binom{20}{10} \cdot \binom{60}{30}$$

■