Solutions to In-Class Problems Week 2, Fri.

Problem 1. Subset take-away[1] is a two player game involving a fixed finite set, A. Players alternately choose proper, nonempty subsets of A with the condition that one may not name a set containing a set that was named earlier. A player who is unable to move loses.

For example, if A is $\{1\}$, then there are no legal moves and the second player wins. If A is $\{1, 2\}$, then the only legal moves are $\{1\}$ and $\{2\}$. Each is a good reply to the other, and so once again the second player wins.

The first interesting case is when A has three elements. This time, if the first player picks a subset with one element, the second player picks the subset with the other two elements. If the first player picks a subset with two elements, the second player picks the subset whose sole member is the third element. Both cases produce positions equivalent to the starting position when A has two elements, and thus leads to a win for the second player.

Verify that when A has four elements, the second player still has a winning strategy.[2]

Solution. Case 1:[1st player chooses a size 1 subset] Then 2nd player should choose the size 3 subset consisting of the other 3 elements. This results in a position corresponding to a size 3 game, and is therefore a win for the 2nd player.

Case 2:[1st player chooses a size 3 subset] Then 2nd player should choose the size 1 subset containing the remaining element. This results in a position corresponding to a size 3 game, and is therefore a win for the 2nd player.

Case 3:[1st player chooses a size 2 subset $\{a, b\}$] Then 2nd player should choose the complementary size 2 subset $\{c, d\}$. Here a, b, c, d are the numbers 1,2,3,4 in some order. So the possible remaining moves are the four other subsets of size 2 and the four subsets of size 1.

> **Subcase 3.1:** At some point in further play, Player 1 picks a size 2 subset, $\{a, c\}$. Then its complement, $\{b, d\}$, will also be available as a possible move, and Player 2 should choose the complement. This leaves a situation in which one of the Subcases 3.1–3 will again apply.
>
> **Subcase 3.2:** At some point in further play, Player 1 picks a size 1 subset, $\{x\}$, and this leaves a size 2 subset available as a legal move. Then a size 2 subset, $\{x, y\}$, must have

[1] From Christenson & Tilford, *David Gale's Subset Takeaway Game, American Mathematical Monthly, Oct. 1997*

[2] David Gale worked out some of the properties of this game and conjectured that the second player wins the game for any set A. This remains an open problem.

been chosen previously, and the size 1 set $\{y\}$ will be available as a possible move. Player 2 should choose $\{y\}$. This leaves a situation that is the same as the game on the complement of $\{x, y\}$, which Player 2 can win.

Subcase 3.3: At some point in further play, Player 1 picks a size 1 subset and no size 2 subset remains as a legal move. Then some other size 1 subset will also available as a possible move, and Player 2 should choose any such size 1 subset. This either leaves no more moves, or the situation of a game of size 2, which Player 2 will win in either case.

We leave it to the reader to check that the moves prescribed in the subcases 3.1–3 will indeed be possible. Granting this, it follows that these subcases exhaust the possible plays, which shows that Player 2 wins in any of the subcases. Hence Player 2 has a winning strategy in all cases. ■

Problem 2. **(a)** Define a bijection between $\mathbb{N}$ and $\mathbb{Z}$.

Solution. One such bijection is

$$f(n) ::= \begin{cases} -n/2 & \text{if } n \text{ is even,} \\ (n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

■

(b) Define a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ (the ordered pairs (0,0),(0,1),(1,2),... of natural numbers).

Solution. Let L_k be the list of pairs whose sum is k in the increasing order of first coordinate, so $L_k = (0, k), (1, k-1), (2, k-2), \ldots, (k, 0)$. Then form the infinite list consisting of the elements of L_0 then the elements of L_1, then the elements of L_2, ...:

$$(0,0), (0,1), (1,0), (0,2)(1,1)(2,0), (0,3)\ldots$$

Define $f(n)$ to be the $n-1$st pair in the list (so $f(0) = (0,0)$). ■