Fallacies with Infinity

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Consider the following claim:

Claim 1. *There exists an infinite decreasing sequence of natural numbers.*

Proof. Assume for sake of contradiction that the longest decreasing sequence of natural numbers is finite. Let $S = \{a_1, a_2, \ldots, a_n\}$ be such a sequence. Then, choose some a_0 larger than $\max_i a_i$, and note that $S' = \{a_0, a_1, a_2, \ldots, a_n\}$ forms a length-$(n+1)$ decreasing sequence. This contradicts the maximality of S, and hence completes the proof. □

Clearly, there must be something wrong with this proof, since *there is no infinite decreasing sequence of natural numbers*! What is the error?

The error lies in the first sentence of the proof: one is not allowed to assume that a longest decreasing sequence exists. In fact, in this case, a longest decreasing sequence does *not* exist. The Well Ordering Principle only allows us to assume that a *shortest* decreasing sequence exists.

A few students attempted proofs similar to this one for parts (a) and (b) of problem 3 of Problem Set 3. These solutions were not valid.

For completeness, let us look at another common error with infinity. We give another "proof" of Claim 1.

Proof. Let $P(n)$ be the statement, "There exists a decreasing length-n sequence of natural numbers."

We prove that $P(n)$ holds for all n by induction. The base case $P(1)$, is clear, as we can just take the single-element sequence $\{1\}$. For the inductive step, suppose that $P(n)$ holds, and let $\{a_1, a_2, \ldots, a_n\}$ be a corresponding decreasing sequence. Then, choose some a_0 larger than $\max_i a_i$, and note that $\{a_0, a_1, a_2, \ldots, a_n\}$ forms a length-$(n+1)$ decreasing sequence. This demonstrates $P(n+1)$ and hence completes the proof. □

What is wrong with this proof? This proof demonstrates the common error of "infinite induction". It is tempting to say that if $P(1)$ holds and if $P(n) \to P(n+1)$ for all $n \in \mathbb{N}$, then $P(\infty)$ holds. However, the principle of induction does not guarantee this. It only guarantees that $P(n)$ holds for all $n \in \mathbb{N}$. In fact, the statement $P(\infty)$ is not always well-defined!

Moral of the story: Be *very* careful when dealing with infinity! When in doubt, run the proof by a TA or by your friend that has taken 18.100B.