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Notes for Recitation 13

1 Asymptotic Notation

Which of these symbols

$$\Theta \qquad O \qquad \Omega \qquad o \qquad \omega$$

can go in these boxes? (List all that apply.)

$$2n + \log n = \boxed{}(n)$$

Θ, O, Ω

$$\log n = \boxed{}(n)$$

O, o

$$\sqrt{n} = \boxed{}(\log^{300} n)$$

Ω, ω

$$n2^n = \boxed{}(n)$$

Ω, ω

$$n^7 = \boxed{}(1.01^n)$$

O, o

2 Asymptotic Equivalence

Suppose $f, g : \mathbb{Z}^+ \to \mathbb{Z}^+$ and $f \sim g$.

1. Prove that $2f \sim 2g$.

   Solution.
   $$\frac{2f}{2g} = \frac{f}{g},$$
   so they have the same limit as $n \to infty$. ■

2. Prove that $f^2 \sim g^2$.

   Solution.
   $$\lim_{n\to\infty} \frac{f(n)^2}{g(n)^2} = \lim_{n\to\infty} \frac{f(n)}{g(n)} \cdot \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f(n)}{g(n)} \cdot \lim_{n\to\infty} \frac{f(n)}{g(n)} = 1 \cdot 1 = 1.$$
   ■

3. Give examples of f and g such that $2^f \not\sim 2^g$.

   Solution.
   $$\begin{aligned} f(n) &= n + 1 \\ g(n) &= n. \end{aligned}$$
   Then $f \sim g$ since $\lim(n + 1)/n = 1$, but $2^f = 2^{n+1} = 2 \cdot 2^n = 2 \cdot 2^g$ so
   $$\lim \frac{2^f}{2^g} = 2 \neq 1.$$
   ■

4. Show that $\sim$ is an equivalence relation

   Solution. (a) Reflexive: $f \sim f$ since $f(x)/f(x) = 1$ for all x (assuming $f(x) \neq 0$), so $\lim_{x\to\infty} f(x)/f(x) = 1$

   (b) Symmetric: $f \sim g$ implies $g \sim f$ since if $\lim_{x\to\infty} f(x)/g(x) = 1$, then by the laws of limits $lim_{x\to\infty} g(x)/f(x) = 1$

   (c) Transitive: $f \sim g$ and $g \sim h$ implies $f \sim h$: if $\lim_{x\to\infty} f(x)/g(x) = 1$, and $\lim_{x\to\infty} g(x)/h(x) = 1$, then multiplying the limits we get
   $$\lim_{x\to\infty} f(x)/h(x) = \lim_{x\to\infty} \frac{f(x)}{g(x)} \times \frac{g(x)}{h(x)} = 1$$
   ■

5. Show that Θ is an equivalence relation

   Solution. (a) Reflexive: $\lim_{x\to\infty} f(x)/f(x) = 1 < \infty$, trivial.

   (b) Symmetric: If $f = \Theta(g)$, we wish to show $g = \Theta(f)$. From the definiton: $\lim_{x\to\infty} f(x)/g(x) = c$ for some non-zero finite constant c. Hence $\lim_{x\to\infty} g(x)/f(x) = 1/c$. Also a non-zero finite constant, so $g = \Theta(f)$.

   (c) Transitive: Want to show $f = \Theta(g), g = \Theta(h)$ then $f = \Theta(h)$. Let $\lim_{x\to\infty} f(x)/g(x) = c_1$ and $\lim_{x\to\infty} g(x)/h(x) = c_2$. Then $\lim_{x\to\infty} f(x)/h(x) = \lim_{x\to\infty} f(x)/g(x) \times g(x)/h(x) = c_1 \times c_2$. Since both c_1 and c_2 are non-zero and finite, so is $c_1 \times c_2$.

   ■

3 More Asymptotic Notation

1. Show that
$$(an)^{b/n} \sim 1.$$
   where a, b are positive constants and $\sim$ denotes asymptotic equality. Hint $an = a2^{\log_2 n}$.

   Solution.
$$(an)^{b/n} = \left(a^b\right)^{1/n} \cdot 2^{(b\log_2 n)/n} \to 1 \cdot 2^0 = 1,$$
   as $n \to \infty$. ■

2. You may assume that if $f(n) \geq 1$ and $g(n) \geq 1$ for all n, then $f \sim g \Rightarrow f^{\frac{1}{n}} \sim g^{\frac{1}{n}}$. Show that
$$\sqrt[n]{n!} = \Theta(n).$$

   Solution.
$$\begin{aligned}
\sqrt[n]{n!} &\sim \left((2\pi n)^{\frac{1}{2}} \left(\frac{n}{e}\right)^n\right)^{1/n} && \text{(Stirling)} \\
&\sim (2\pi n)^{\frac{1}{2n}} \frac{n}{e} \\
&\sim 1 \cdot \frac{n}{e} && \text{part (a)} \\
&= \Theta(n)
\end{aligned}$$

   ■