Massachusetts Institute of Technology
6.042J/18.062J, Spring '15: Mathematics for Computer Science — April 6
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# In-Class Problems Week 8, Mon.

**Problem 1.**
For each of the binary relations below, state whether it is a strict partial order, a weak partial order, an equivalence relation, or none of these. If it is a partial order, state whether it is a linear order. If it is none, indicate which of the axioms for partial-order and equivalence relations it violates.

**(a)** The superset relation $\supseteq$ on the power set pow $\{1, 2, 3, 4, 5\}$.

**(b)** The relation between any two nonnegative integers $a$ and $b$ such that $a \equiv b \pmod 8$.

**(c)** The relation between propositional formulas $G$ and $H$ such that $[G \text{ IMPLIES } H]$ is valid.

**(d)** The relation between propositional formulas $G$ and $H$ such that $[G \text{ IFF } H]$ is valid.

**(e)** The relation 'beats' on Rock, Paper, and Scissors (for those who don't know the game Rock, Paper, Scissors, Rock beats Scissors, Scissors beats Paper, and Paper beats Rock).

**(f)** The empty relation on the set of real numbers.

**(g)** The identity relation on the set of integers.

**(h)** The divisibility relation on the integers, $\mathbb{Z}$.

**Problem 2.**
The proper subset relation, $\subset$, defines a strict partial order on the subsets of $[1..6]$, that is, on pow$([1..6])$.

**(a)** What is the size of a maximal chain in this partial order? Describe one.

**(b)** Describe the largest antichain you can find in this partial order.

**(c)** What are the maximal and minimal elements? Are they maximum and minimum?

**(d)** Answer the previous part for the $\subset$ partial order on the set pow $[1..6] - \emptyset$.

**Problem 3.**
Let $S$ be a sequence of $n$ different numbers. A *subsequence* of $S$ is a sequence that can be obtained by deleting elements of $S$.

For example, if

$$S = (6, 4, 7, 9, 1, 2, 5, 3, 8)$$

Then 647 and 7253 are both subsequences of $S$ (for readability, we have dropped the parentheses and commas in sequences, so 647 abbreviates $(6, 4, 7)$, for example).

An *increasing subsequence* of $S$ is a subsequence of whose successive elements get larger. For example, 1238 is an increasing subsequence of $S$. Decreasing subsequences are defined similarly; 641 is a decreasing subsequence of $S$.

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**(a)** List all the maximum-length increasing subsequences of $S$, and all the maximum-length decreasing subsequences.

Now let $A$ be the *set* of numbers in $S$. (So $A$ is the integers $[1..9]$ for the example above.) There are two straightforward linear orders for $A$. The first is numerical order where $A$ is ordered by the $<$ relation. The second is to order the elements by which comes first in $S$; call this order $<_S$. So for the example above, we would have

$$6 <_S 4 <_S 7 <_S 9 <_S 1 <_S 2 <_S 5 <_S 3 <_S 8$$

Let $\prec$ be the product relation of the linear orders $<_s$ and $<$. That is, $\prec$ is defined by the rule

$$a \prec a' \quad ::= \quad a < a' \text{ AND } a <_S a'.$$

So $\prec$ is a partial order on $A$ (Section 9.9 in the course textbook).

**(b)** Draw a diagram of the partial order, $\prec$, on $A$. What are the maximal and minimal elements?

**(c)** Explain the connection between increasing and decreasing subsequences of $S$, and chains and anti-chains under $\prec$.

**(d)** Prove that every sequence, $S$, of length $n$ has an increasing subsequence of length greater than $\sqrt{n}$ or a decreasing subsequence of length at least $\sqrt{n}$.

**Problem 4.**

For any total function $f : A \to B$ define a relation $\equiv_f$ by the rule:

$$a \equiv_f a' \quad \text{iff} \quad f(a) = f(a'). \tag{1}$$

**(a)** Observe (and sketch a proof) that $\equiv_f$ is an equivalence relation on $A$.

**(b)** Prove that every equivalence relation, $R$, on a set, $A$, is equal to $\equiv_f$ for the function $f : A \to \text{pow}(A)$ defined as

$$f(a) ::= \{a' \in A \mid a \; R \; a'\}.$$

That is, $f(a) = R(a)$.

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