In-Class Problems Week 13, Mon.

Guess the Bigger Number Game

Team 1:

- Write two different integers between 0 and 7 on separate pieces of paper.
- Put the papers face down on a table.

Team 2:

- Turn over one paper and look at the number on it.
- Either stick with this number or switch to the other (unseen) number.

Team 2 wins if it chooses the larger number; else, Team 1 wins.

Problem 1.
The analysis given before class implies that Team 2 has a strategy that wins 4/7 of the time no matter how Team 1 plays. Can Team 2 do better? The answer is "no," because Team 1 has a strategy that guarantees that it wins at least 3/7 of the time, no matter how Team 2 plays. Describe such a strategy for Team 1 and explain why it works.

Problem 2.
Let I_A and I_B be the indicator variables for events A and B. Prove that I_A and I_B are independent iff A and B are independent.

Hint: Let $A^1 ::= A$ and $A^0 ::= \overline{A}$, so the event $[I_A = c]$ is the same as A^c for $c \in \{0, 1\}$; likewise for B^1, B^0.

Problem 3.
Let $R_1, R_2, \ldots, R_m$, be mutually independent random variables with uniform distribution on $[1, n]$. Let $M ::= \max\{R_i \mid i \in [1, m]\}$.

(a) Write a formula for $\text{PDF}_M(1)$.

(b) More generally, write a formula for $\Pr[M \leq k]$.

(c) For $k \in [1, n]$, write a formula for $\text{PDF}_M(k)$ in terms of expressions of the form "$\Pr[M \leq j]$" for $j \in [1, n]$.

Problem 4.
Suppose you have a biased coin that has probability p of flipping heads. Let J be the number of heads in n independent coin flips. So J has the general binomial distribution:

$$\mathrm{PDF}_J(k) = \binom{n}{k} p^k q^{n-k}$$

where $q ::= 1 - p$.

(a) Show that

$$\begin{aligned} \mathrm{PDF}_J(k-1) &< \mathrm{PDF}_J(k) && \text{for } k < np + p, \\ \mathrm{PDF}_J(k-1) &> \mathrm{PDF}_J(k) && \text{for } k > np + p. \end{aligned}$$

(b) Conclude that the maximum value of PDF_J is asymptotically equal to

$$\frac{1}{\sqrt{2\pi npq}}.$$

Hint: For the asymptotic estimate, it's ok to assume that np is an integer, so by part (a), the maximum value is $\mathrm{PDF}_J(np)$. Use Stirling's Formula.

Supplemental problem

Problem 5.
You have just been married and you both want to have children. Of course, any child is a blessing, but your spouse prefers girls, so you decide to keep having children until you have a girl. In other words, if your 1st child is a girl, you'll stop there. If it's a boy, then you'll have a 2nd child, too. If that one is a girl, you'll stop there. Otherwise, you'll have a 3rd child, and so on. Assume that you will never abandon this ingenious plan and that every child is equally likely to be a boy or a girl, independently of the number of its brothers so far. Let B be the *boys* that you will eventually have to put up with to enjoy the company of your beloved daughter.

(a) For $i = 0, 1, 2, \ldots$, what is the value of $\mathrm{PDF}_B(i)$?

(b) For $i = 0, 1, 2, \ldots$, what is the value of $\mathrm{CDF}_B(i)$?