



Mathematics for Computer Science
MIT 6.042J/18.062J

Linearity of Expectation



Albert R Meyer, May 8, 2013 ranvarexpectlin.1




Linearity of Expectation


R, S random variables, a, b constants

$$E[aR + bS] = aE[R] + bE[S]$$

even if R, S are dependent



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Linearity of Expectation


R, S random variables, a, b constants

$$E[aR + bS] = aE[R] + bE[S]$$

proof by rearranging terms in the defining sum




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Linearity of Expectation

$$E[aA + bB] = \sum_{\omega} (aA(\omega) + bB(\omega)) \cdot \Pr[\omega] = a(\sum_{\omega} A(\omega) \cdot \Pr[\omega]) + b(\sum_{\omega} B(\omega) \cdot \Pr[\omega]) = aE[A] + bE[B]$$

QED




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
Expectation of indicator I_A

$$E[I_A] ::= 1 \cdot \Pr[I_A=1] + 0 \cdot \Pr[I_A=0]$$

$$= \Pr[I_A=1]$$


$$= \Pr[A]$$


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Expected #Heads in n Flips

H_i is indicator for Head on i^{th} flip

$$\#H's = H_1 + H_2 + \dots + H_n$$


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Expected #Heads


$$E[\#H's] = E[H_1 + H_2 + \dots + H_n]$$

so by linearity

$$= E[H_1] + E[H_2] + \dots + E[H_n]$$

$$= n \cdot \Pr[\text{Head}] = np$$



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Expected #hats returned

n men each check their hat.
Hats get scrambled so
 $\Pr[i^{\text{th}} \text{ man gets own hat back}] = 1/n$

How many men do we expect will get their hat back?



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6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Expected #hats returned

R_i indicates i^{th} man got his hat returned.

Notice R_i and R_j are **not** independent!



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6	9	13	7
12	10	5	
3	1	4	14
15	8	11	2

Expected #hats returned

R_i indicates i^{th} man got his hat returned. Even so

$$\begin{aligned} E[\# \text{ hats returned}] &= \\ E[R_1 + R_2 + \dots + R_n] &= \\ E[R_1] + E[R_2] + \dots + E[R_n] &= \\ = n(1/n) &= 1 \end{aligned}$$



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6	9	13	7
12	10	5	
3	1	4	14
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Chinese Banquet

Say n people sit around a spinner (a "lazy-Susan") with n different dishes.

Spin randomly.

How many people do we **expect** will get same dish as initially?



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6	9	13	7
12	10	5	
3	1	4	14
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Chinese Banquet

Now R_i indicates i^{th} person got same dish

R_i 's are totally **dependent**

— all **1** or all **0**

but **linearity still holds**

Expectation still is **1**



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6	9	13	7
12	10	5	
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15	8	11	2

Independent Product of Expectations

For independent X, Y

$$E[X \cdot Y] = E[X] \cdot E[Y]$$

proof by rearranging terms in the defining sum again



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6	9	13	7
12	10	5	
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Independent Product of Expectations

$$\begin{aligned} E[XY] &::= \sum_{x,y} xy \Pr[X=x \text{ AND } Y=y] \\ (\text{indep}) &= \sum_{x,y} xy \Pr[X=x] \Pr[Y=y] \\ &= \sum_y \sum_x xy \Pr[X=x] \Pr[Y=y] \end{aligned}$$



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6	9	13	7
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Independent Product of Expectations

$$\begin{aligned} E[XY] &::= \sum_{x,y} xy \Pr[X=x \text{ AND } Y=y] \\ (\text{indep}) &= \sum_{x,y} xy \Pr[X=x] \Pr[Y=y] \\ &= \sum_y \sum_x xy \Pr[X=x] \Pr[Y=y] \end{aligned}$$



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6	9	13	7
12	10	5	
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Independent Product of Expectations

$$\begin{aligned} E[XY] &::= \sum_{x,y} xy \Pr[X=x \text{ AND } Y=y] \\ (\text{indep}) &= \sum_{x,y} xy \Pr[X=x] \Pr[Y=y] \\ &= \sum_y \sum_x xy \Pr[X=x] \Pr[Y=y] \\ &= \sum_y (y \Pr[Y=y]) \sum_x x \Pr[X=x] \end{aligned}$$




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
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Independent Product of Expectations

$$\begin{aligned}
 E[XY] &::= \sum_{x,y} xy \Pr[X=x \text{ AND } Y=y] \\
 (\text{indep}) &= \sum_{x,y} xy \Pr[X=x] \Pr[Y=y] \\
 &= \sum_y \sum_x xy \Pr[X=x] \Pr[Y=y] \\
 &= \sum_y (y \Pr[Y=y] \sum_x x \Pr[X=x]) \\
 &= (\sum_x x \Pr[X=x]) (\sum_y y \Pr[Y=y]) \\
 &= E[X]E[Y]
 \end{aligned}$$



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Independent Product of Expectations

$$\begin{aligned}
 E[XY] &::= \sum_{x,y} xy \Pr[X=x \text{ AND } Y=y] \\
 (\text{indep}) &= \sum_{x,y} xy \Pr[X=x] \Pr[Y=y] \\
 &= \sum_y \sum_x xy \Pr[X=x] \Pr[Y=y] \\
 &= \sum_y (y \Pr[Y=y] \sum_x x \Pr[X=x]) \\
 &= (\sum_x x \Pr[X=x]) (\sum_y y \Pr[Y=y]) \\
 &= E[X]E[Y] \quad \text{QED}
 \end{aligned}$$



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Blunders

Don't assume product rule without independence.

Example: Say X takes positive and negative values with equal probability. So

$$E[X] = 0 < E[X^2]$$



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Blunders

Don't assume product rule without independence.

Example: Say X takes positive and negative values with equal probability. So

$$E[X] \cdot E[X] = 0 < E[X^2]$$


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6	9	13	7
12	10	5	
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Blunders

Don't assume a reciprocal expectation rule. In general,

$$E\left[\frac{X}{Y}\right] \neq \frac{E[X]}{E[Y]}$$

even with independence



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