Equivalence relations are another kind of binary relation on a set which play a crucial role in mathematics and in computer science in particular. And they can also be explained both in terms of digraphs and in terms of axioms.

So let's begin with a digraph explanation of an equivalence relation. And the kind of relation that's an equivalence relation is the relation of there being a walk in both directions between two vertices. So if there's a walk between vertex u and vertex v and conversely there's a walk from vertex v back to vertex u, then u and v are said to be strongly connected, and strongly connected is going to be an example of an equivalence relation. So in terms of the walk relation, including 0-length walks, the relation we're talking about is u G* v and v G* u.

Now, as a property of relations, this has a name. It's called symmetry. So a relation R on a set A is symmetric if and only if a R b implies b R a, and the first remark is that the strongly connected relation is symmetric.

An equivalence relation is a symmetric relation that is transitive and reflexive. And again, we have immediately that the walk relation-- the mutual walk relation, the two-way walk relation or strongly connected relation in a digraph is an equivalence relation. Because clearly if there's two way paths between u and v and between and v and w, then there's one between u and w by going for u to v to w and back. Likewise, there is a length 0 walk from any element to itself. And by definition, strong connectedness is symmetric.

So the strong connectedness relation in any digraph is an equivalence relation. And the theorem is, conversely, that any equivalence relation, anything that's an equivalence relation, is the strongly connected relation of some digraph. The proof is trivial. It's the strongly connected relation of itself.

OK. Some examples of equivalence relations to see why they're so basic is that the most fundamental one is equality. Obviously, equality is symmetric and reflexive and transitive, and so it's an equivalence relation. Another one that we've seen is congruence mod n, which you could also check is symmetric and transitive and reflexive. And finally, another relation would be that two sets are the same size, providing they're finite sets. And another example would be a bunch of objects having the same color. Two objects have the same color is a relation among objects that have color that is symmetric and transitive and reflexive, so it's an equivalence relation.

Let's illustrate some of these axioms that we have in terms of graphs. It can be helpful to remember them.

So reflexive means that when you look at a digraph, it's reflexive when there's a little self loop from every vertex to itself. So there's a length 1 path or an edge from vertex to itself in reflexive graphs.

Transitive means that whenever you have two edges connecting one vertex to another, there's a path of length 2 from one place to another that in fact is an edge from that place to its target. And of course as we said, once there is an edge wherever there's a path of length 2, it follows by induction that there's an edge wherever there's a path of any length, and that's what transitive means.

Asymmetric means that whenever you have an edge from one vertex to another there is no edge back. So in particular, if I have an edge from this vertex to that vertex in blue, there is no edge that goes back in the other direction. Nor is there ever a self loop in an asymmetric graph.

And finally, in a symmetric graph, wherever there's an edge, there's an edge that goes back the other way.

So that can help you maybe remember what these properties mean.

Now again, equivalence relations, besides being represented in terms of the strongly connected relation of a digraph, can be represented in two other very natural ways that really explains where they come from and what their properties are.

So whenever you have a total function f on a set A, it defines an equivalence relation on the set A. Namely, if f is a total function from domain A to codomain B, then we can define a relation we can call equivalence sub f on the set A by the rule that two elements are equivalents of f if and only if they have the same image under f-- they hit the same thing. That is, A is equivalent sub f to A prime if and only if f of a is equal to f of a prime. And again, equivalence sub f immediately inherits the properties of equality, which makes it an equivalence relation.

And the theorem that we have is that every relation R on a set A is an equivalence relation if and only if it in fact is equal to equivalence sub f for some function f.

Let's illustrate that. We already remembered that congruence mod n can be understood as equivalence sub f, where the mapping is just map things to remainders. Two numbers are congruent mod n if and only if they have the same remainder on division by n. So map a number a to f of k, equal its remainder, and we have found the equivalence sub f representation of congruence, which is another way to verify that congruence is an equivalence relation.

Finally, whenever you have a partition of a set, you can define an equivalence relation. So a partition of a set cuts up the set A into a bunch of blocks which are nonempty, and every element is a member of some block, and the blocks don't overlap. So in fact, every element is a member of a unique block. And that enables me to define an equivalence relation on A by the property that two elements are in the same block.

In fact, that's the proof of the previous representation theorem in terms of a function that you can map an element to the block that it's in, in order to see that the block representation and the equivalence sub f representation are the same. The proof in the other direction, that every equivalence relation can be represented in this way, is an exercise in axiomatic reasoning, and elementary one that we're going to leave to a problem and not do in this presentation.

So the theorem finally is that, again, a relation R on a set is an equivalence relation if and only if it is in fact the being in the same block relation for some partition.

And that is the story and multiple ways of understanding what equivalence relations are.