6.055J / 2.038J The Art of Approximation in Science and Engineering
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Therefore, planes and well-fed, migrating birds should have the same maximum range! Let's check. The longest known nonstop flight by an animal is 11,570 km, made by a bar-tailed godwit from Alaska to New Zealand (tracked by satellite). The maximum range for a 747-400 is 13,450 km, only slightly longer than the godwit's range.

6.2.4 Explicit computations

To get an explicit range, not only how the range scales with size, estimate the fuel fraction β, the energy density $\mathcal{E}$, and the drag coefficient C. For the fuel fraction I'll guess $\beta \sim 0.4$. For $\mathcal{E}$, look at the nutrition label on the back of a pack of butter. Butter is almost all fat, and one serving of 11 g provides 100 Cal (those are 'big calories'). So its energy density is $9\,\mathrm{kcal\,g^{-1}}$. In metric units, it is $4 \cdot 10^7\,\mathrm{J\,kg^{-1}}$. Including a typical engine efficiency of one-fourth gives

$$\mathcal{E} \sim 10^7\,\mathrm{J\,kg^{-1}}.$$

The modified drag coefficient needs converting from easily available data. According to Boeing, a 747 has a drag coefficient of $C' \approx 0.022$, where this coefficient is measured using the wing area:

$$F_{\mathrm{drag}} = \frac{1}{2} C' A_{\mathrm{wing}} \rho v^2.$$

Alas, this formula is a third convention for drag coefficients, depending on whether the drag is referenced to the cross-sectional area A, wing area A_{wing}, or squared wingspan L^2.

It is easy to convert between the definitions. Just equate the standard definition

$$F_{\mathrm{drag}} = \frac{1}{2} C' A_{\mathrm{wing}} \rho v^2.$$

to our definition

$$F_{\mathrm{drag}} = C L^2 \rho v^2$$

to get

$$C = \frac{1}{2} \frac{A_{\mathrm{wing}}}{L^2} C' = \frac{1}{2} \frac{l}{L} C',$$

since $A_{\mathrm{wing}} = Ll$ where l is the wing width. For a 747, $l \sim 10\,\mathrm{m}$ and $L \sim 60\,\mathrm{m}$, so $C \sim 1/600$.

Combine the values to find the range:

$$s \sim \frac{\beta \mathcal{E}}{C^{1/2} g} \sim \frac{0.4 \times 10^7\,\mathrm{J\,kg^{-1}}}{(1/600)^{1/2} \times 10\,\mathrm{m\,s^{-2}}} \sim 10^7\,\mathrm{m} = 10^4\,\mathrm{km}.$$

The maximum range of a 747-400 is 13,450 km. The maximum known nonstop flight by a bird – indeed, by any animal – is 11,570 km: A female bar-tailed godwit tracked by satellite migrated between Alaska and New Zealand. The approximate analysis of the range is unreasonably accurate.

Next I estimate the minimum-energy speed and compare it to the cruising speed of a 747. The sum of drag and lift energies is a minimum when the speed is given by

$$Mg \sim C^{1/2}\rho v^2 L^2.$$

The speed is

$$v \sim \left(\frac{Mg}{C^{1/2}\rho L^2}\right)^{1/2}.$$

A fully loaded 747 has $M \sim 4 \cdot 10^5$ kg. The drag coefficient is again $C \sim 1/600$, the wingspan is $L \sim 60$ m, and the air density up high is $\rho \sim 0.5\,\mathrm{kg\,m^{-3}}$. So

$$v \sim \left(\frac{4 \cdot 10^5\,\mathrm{kg} \times 10\,\mathrm{m\,s^{-2}}}{(1/600)^{1/2} \times 0.5\,\mathrm{kg\,m^{-3}} \times 3.6 \cdot 10^3\,\mathrm{m^2}}\right)^{1/2}.$$

Do the arithmetic mentally. The $\sqrt{1/600}$ in the denominator becomes a 25 in the numerator. Combined with the $4 \cdot 10^5$, it becomes 10^7. Including the 10 from g, the numerator is 10^8 and the denominator is roughly $2 \cdot 10^3$, so

$$v \sim \left(\frac{1}{2} \cdot 10^5\right)^{1/2}\,\mathrm{m\,s^{-1}} = 5^{1/2} \times 100\,\mathrm{m\,s^{-1}} \sim 220\,\mathrm{m\,s^{-1}}.$$

That speed is roughly 500 mph, reasonably close to the 747's maximum speed of 608 mph.