6.055J / 2.038J The Art of Approximation in Science and Engineering
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To find C, notice that there are six ways to get an abc factor. So $C = 6$ and then $B = 3$.

Thus

$$(a + b + c)^3 = (a^3 + b^3 + c^3) + 3(a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2) + 6abc.$$

This symmetry solution has several merits. First, it is less prone to mistakes than is multiplying by brute force. Second, it produces the answer in a meaningful, low-entropy form. The chunks in the solution – the terms $a^3 + b^3 + c^3$ and $a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2$ and abc – each obey the symmetry that nothing important changes if you permute a, b, and c. Rather than using a brute-force method and then doing hard work to turn the solution into a meaningful form, use symmetry reasoning: Whenever possible, work with quantities that obey the symmetries of the problem.

This chapter shows how this idea leads naturally to dimensionless groups, the fundamental idea of dimensional analysis.

7.1 Power of multinationals

The first example shows what happens when people take no notice of dimensions.

Critics of globalization often make this argument:

> In Nigeria, a relatively economically strong country, the GDP [gross domestic product] is $99 billion. The net worth of Exxon is $119 billion. 'When multinationals have a net worth higher than the GDP of the country in which they operate, what kind of power relationship are we talking about?' asks Laura Morosini. [Source: 'Impunity for Multinationals', *ATTAC*, 11 Sept 2002, [url:nigeria-argument], retrieved 11 Sept 2006]

Before reading further, try to find the most egregious fault in the comparison between Exxon and Nigeria. It's a competitive field, but one fault stands out.

The comparison between Exxon and Nigeria has many problems. First, the comparison exaggerates Exxon's power by using its worldwide assets (net worth) rather than its assets only in Nigeria. On the other hand, Exxon can use its full international power when negotiating with Nigeria, so perhaps the worldwide assets are a fair basis for comparison.

A more serious, and less debatable, problem is the comparison with GDP, or gross domestic product. To see the problem, look at the ingredients in how GDP is usually measured: as dollars per year. The $99 billion for Nigeria's GDP is shorthand for $99 billion per year. A year is an astronomical time, and its use in an economic measurement is arbitrary. Economic flows, which are a social phenomenon, should not care about how long the earth requires to travel around the sun. Suppose instead that the decade was the chosen unit of time in measuring the GDP. Then Nigeria's GDP would be roughly $1 trillion per decade (assuming that the $99 billion per year value held steady) and would be reported as $1 trillion. Now Nigeria towers over the puny Exxon whose assets are a mere one-tenth of this figure.

To produce the opposite conclusion, just measure GDP in units of dollars per week: Nigeria's GDP becomes $2 billion per week. Now puny Nigeria stands helpless before the might of Exxon, 50-fold larger than Nigeria. Either conclusion about the relative powers can be produced merely by changing the units. This arbitrariness indicates that the comparison is bogus.

The flaw in the comparison is the theme of this chapter. Assets, or net worth, are an amount of money – money is its dimensions – and are typically measured in units of dollars. GDP is defined as the total goods and services sold in one year. It is a rate and has dimensions of money per time; its typical units are dollars per year. Comparing assets to GDP means comparing money to money per time. *Because the dimensions of these two quantities are not the same, the comparison is nonsense!* A similarly flawed comparison is to compare length per time (speed) with length. Listen how ridiculous it sounds: 'I walk 1.5 meters per second, much smaller than the Empire State building in New York, which is 300 meters high.' To produce the opposite conclusion, measure time in hours: 'I walk 5000 meters per hour, much larger than the Empire State building at only 300 meters.' Nonsense all around!

This example illustrates several ideas:

- *Dimensions versus units.* Dimensions are general and generic, such as money per time or length per time. Units are the instantiation of dimensions in a system of measurement. The most complete system of measurement is the System International (SI), where the unit of mass is the kilogram, the unit of time the second, and the unit of length the meter. Other examples of units are dollars per year or kilometers per year.

- *Necessary condition for a valid comparison.* In a valid comparison, the dimensions of the compared objects be identical. Do not compare apples to oranges (except in questions of taste, like 'I prefer apples to oranges.')

- *Rubbish abounds.* There's lots of rubbish out there, so keep your eyes open for it!

- *Bad argument, fine conclusion.* I agree with the conclusion of the article, that large oil companies exert massive power over poor countries. However, as a physicist I am embarrassed by the reasoning. This example teaches me a valuable lesson about theorems and proofs: judge the proof not just the theorem. Even if you disagree with the conclusion, remember the general lesson that a correct conclusion does not validate a dubious argument.

7.2 Pyramid volume

The last example showed the value of dimensions in economics. The next example shows that dimensions are also useful in mathematics. What is the volume of this square-based pyramid? Here are several choices:

1. $\frac{1}{3}bh$

2. $b^3 + h^2$
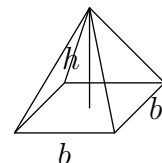
3. b^4/h

4. bh^2

Let's take the choices in turn. The first choice, $bh/3$, has dimensions of area rather than volume. So it cannot be right. The second choice, $b^3 + h^2$, begins with a volume in the b^3 term but falls apart with the h^2, which has dimensions of area. Since it adds an area to a volume – the crime of dimension mixing – it cannot be right. The third choice, b^4/h, has dimensions of volume, so it might be correct. It even increases as b increases, which is a good sign. However, the volume should increase as h increases – a proportional-reasoning argument – whereas this choice indicates that the volume decreases as h increases! So it cannot be right.

The final choice, bh^2, has correct dimensions and increases as h or b increases. Does it increase by the right amounts? Imagine drilling into the pyramid from the top and dividing it into thin cores or volume elements. If the height of the pyramid doubles, then each vertical volume element doubles in volume; so the volume of the pyramid should double. In symbols, $V \propto h$. But bh^2 quadruples when h doubles, so that choice cannot be right.

The requirement that $V \propto h$ together with the requirement that V have dimensions of length cubed means that the missing item in $V \propto h$ is an area. The only way to make an area from b is to make b^2 perhaps times a dimensionless constant. So

$$V \sim hb^2.$$

The missing dimensionless constant is hidden in the twiddle $\sim$ sign. Alternatively, the ratio V/hb^2 is dimensionless.

This method of deducing the volume requires remembering hardly any arbitrary data. It requires these ingredients:

1. Using vertical volume elements to find out that $V \propto h$.

2. Using dimensions along with $V \propto h$ to show that $V \sim hb^2$.

3. Remembering the correct dimensionless constant.

The first two steps are logic and do not require arbitrary data. Instead they use reasoning methods that you use elsewhere (so there's no marginal cost to remember them). The third step requires seemingly arbitrary data. However, in **Chapter 8** on special cases, I'll show you how to determine the constant elegantly without even needing an integral.

Then the volume requires no memory. Arbitrary data is, by definition, impossible to compress. Dimensions, and more generally our techniques for handling complexity, are a form of data compression or entropy reduction [5]. One way to look at learning is as data compression. So dimensions, and our other techniques, enhance learning.

There's an old saying: Tell the truth; there's less to remember. The similar moral here is: *Use dimensions (and proportional reasoning); there's less to remember!*