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6.055J / 2.038J The Art of Approximation in Science and Engineering  
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# Approximating logarithms using musical intervals

| Semitones | Interval      | Ratio                    | Exact Value |
|-----------|---------------|--------------------------|-------------|
| 2         | M2            | 9/8                      | 1.122       |
| 3         | m3            | 6/5                      | 1.1885      |
| 4         | M3            | 5/4                      | 1.259       |
| 5         | P4            | 4/3                      | 1.3335      |
| 6         | d5            | $\sqrt{2}$               | 1.4125      |
| 7         | P5            | 3/2                      | 1.496       |
| 8         | m6 = P8 - M3  | 8/5                      | 1.585       |
| 9         | M6 = P8 - m3  | 5/3                      | 1.679       |
| 10        | P5 + m3       | 9/5                      | 1.7783      |
|           | 2 · P4        | 16/9                     | 1.7783      |
| 11        |               | 17/9                     | 1.8836      |
| 12        | P8            | 2                        | 1.9953      |
| 17.4      |               | $e$                      | 2.718       |
| 19        | P8 + P5       | 3                        | 2.9854      |
| 24        | 2 · P8        | 4                        | 3.981       |
| 28        | 2 · P8 + M3   | 5                        | 5.012       |
| 31        | 2 · P8 + P5   | 6                        | 5.9566      |
| 34        | 3 · P8 - M2   | $\frac{64}{9} \approx 7$ | 7.080       |
| 36        | 3 · P8        | 8                        | 7.943       |
| 38        | 2 · (P8 + P5) | 9                        | 8.913       |
| 40        | 3 · P8 + M3   | 10                       | 10.         |

| KEY    |                |             |
|--------|----------------|-------------|
| Symbol | Interval       | Notes       |
| M2     | Major 2nd      | C-D         |
| m3     | Minor 3rd      | C-E $\flat$ |
| M3     | Major 3rd      | C-E         |
| P4     | Perfect 4th    | C-F         |
| d5     | Diminished 5th | C-G $\flat$ |
| P5     | Perfect 5th    | C-G         |
| m6     | Minor 6th      | C-A $\flat$ |
| M6     | Major 6th      | C-A         |

The starting point is  $2^{10} \approx 10^3$ , or  $2^{1/12} \approx 10^{1/40}$ . By chance  $2^{1/12}$  is the semitone frequency ratio on the equal-tempered scale. Since we know what Pythagorean ratios the equal-tempered intervals are supposed to approximate, we can approximate logarithms to the base  $2^{1/12}$ , and thereby approximate logarithms to the base  $10^{1/40}$ , which gives us twice the number of decibels. The ratio column indicates the ratios for perfect Pythagorean intervals, and the exact value column shows  $10^{\text{semitones}/40}$ , to show the accuracy of the method. Note that 10 semitones has two possible breakdowns into intervals, as P5 + m3 or 2 · P4. The second is much more accurate, because in the equal-tempered scale, the perfect intervals come out almost exactly right, at the cost of some error in the major and minor intervals.

To use the table to compute  $\log_{10} x$ , find  $x$  as a product of ratios, add the number of semitones for the ratios, and divide by 40 (divide by 2 to get dB). To calculate  $10^x$ , multiply  $x$  by 40, find that value in the semitones column, and read off the corresponding ratio. From a few basic Pythagorean ratios and number of semitones, most of the table is easy to figure out. The most important to remember one is the fifth: 7 semitones corresponds to 3/2. For example, from the fifth we can compute the frequency ratio for a fourth (5 semitones). The two intervals together make an octave, so the product of their frequency ratios is 2. This means 5 semitones corresponds to 4/3. Many other entries can be worked out similarly.

Some examples (arrows point from the real to the log world):

$$2 \rightarrow 1 \text{ octave} = 12 \text{ semitones} = 6 \text{ dB} = 0.3 \text{ decades.}$$

$$\left(\frac{4}{3}\right)^{10} \rightarrow 10 \cdot \text{P4} = 50 \text{ semitones} = 40 \text{ semitones} + 2 \cdot \text{P4} \leftarrow 10 \cdot \frac{16}{9} = 17.78 \text{ (exact 17.76).}$$

$$5 = \frac{5}{4} \cdot 2 \cdot 2 \rightarrow \text{M3} + 2 \cdot \text{P8} = 28 \text{ semitones} = \frac{28}{40} \text{ or } 0.7 \text{ decades (14 dB).}$$

$$3^{10} \rightarrow 10 \cdot (\text{P8} + \text{P5}) = 190 \text{ semitones} = 200 - 2 \cdot \text{P4} \leftarrow 10^{200/40} \cdot \frac{9}{16} = 56250 \text{ (exact 59049).}$$

$$e^{10} \rightarrow 10 \cdot 17.4 \text{ semitones} = 174 \text{ semitones} = 160 + 12 + 2 \text{ semitones} \leftarrow 10^4 \cdot 2 \cdot \frac{9}{8} = 22500.$$

(This method is due to the statistician I. J. Good, who credits his father.)