Lectures on Dynamic Systems and Control

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Chapter 24

Observability

24.1   Introduction

Observability is a notion that plays a major role in filtering and reconstruction of states from inputs and outputs. Together with reachability, observability is central to the understanding of feedback control systems.

24.2   Observability

It turns out it is more natural to think in terms of "unobservability" as reflected in the following definition.

Definition 24.1 A state q of a finite dimensional dynamic system is said to be *unobservable* over $[0, T)$ if, with $x(0) = q$ and for every $u(t)$ over $[0, T)$, we get the same $y(t)$ as we would with $x(0) = 0$, *i.e.* an unobservable initial condition cannot be distinguished from the zero initial condition. The *dynamic system* is called *unobservable* if it has an unobservable state, and otherwise it is called *observable*.

The initial state $x(0)$ can be uniquely determined from input/output measurements iff the system is observable (prove this). This can be taken as an alternate definition of observability.

24.3   Discrete-Time Analysis

We begin with the system description in state space:

$$\begin{aligned}
\mathbf{x}(k+1) &= A\mathbf{x}(k) + B\mathbf{u}(k) \\
\mathbf{y}(k) &= C\mathbf{x}(k) + D\mathbf{u}(k)
\end{aligned} \tag{24.1}$$

Suppose we are given $\mathbf{u}(t)$ and $\mathbf{y}(t)$ for $0 \le t < T$. We can expand (24.1) as follows:

$$\begin{bmatrix} \mathbf{y}(0) \\ \mathbf{y}(1) \\ \vdots \\ \mathbf{y}(T-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{T-1} \end{bmatrix} \mathbf{x}(0)$$

$$+ \begin{bmatrix} D & 0 & 0 & \cdots & 0 \\ CB & D & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ CA^{T-2}B & CA^{T-3}B & 0 & \cdots & D \end{bmatrix} \begin{bmatrix} \mathbf{u}(0) \\ \mathbf{u}(1) \\ \vdots \\ \mathbf{u}(T-1) \end{bmatrix} \tag{24.2}$$

Now the second term on the right — the forced response — is known, so we can subtract it from the vector of measured outputs to get

$$\underline{\mathbf{y}} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{T-1} \end{bmatrix} \mathbf{x}(0) = \mathcal{O}_T \mathbf{x}(0), \tag{24.3}$$

where we have made the obvious definitions for $\underline{\mathbf{y}}$ and the T-step *observability matrix* $\mathcal{O}_T$. The issue of observability over T steps then boils down to our ability to determine $x(0)$ uniquely from knowledge of $\underline{\mathbf{y}}$. Equation (24.3) shows that we only need to check observability for $\mathbf{u} \equiv 0$; the effect of a nonzero input is just to change what $\underline{\mathbf{y}}$ is, but in either case $\underline{\mathbf{y}}$ is a known vector. The following result is an immediate consequence of (24.3).

Theorem 24.1 The set of states that is unobservable over T steps is precisely $\mathcal{N}(\mathcal{O}_T)$, and is therefore a *subspace*.

Notice also that

$$\mathcal{N}(\mathcal{O}_k) \supseteq \mathcal{N}(\mathcal{O}_{k+1}) \tag{24.4}$$
$$\mathcal{N}(\mathcal{O}_n) = \mathcal{N}(\mathcal{O}_{n+\ell}), \quad \ell > 1 \tag{24.5}$$

Equations (24.4) and (24.5) lead to the following theorem.

Theorem 24.2 If $\mathbf{x}(0) = \xi$ is unobservable over n steps, then it is unobservable over any number of steps. Equivalently, the system is observable if and only if $\text{rank}(\mathcal{O}_n) = n$.

Proof: The proofs of all of these results parallel the proofs of similar results in reachability, and are left to the reader to complete.

Note that in the context of reachability, it was the set of reachable states that formed a subspace, whereas now it is the set of *unobservable* states that forms a subspace. We denote this subspace by $\overline{\mathbb{O}}(C, A)$ or simply $\overline{\mathbb{O}}$. It is evident that

$$\overline{\mathbb{O}}(C, A) = \mathbb{R}^{\perp}(A', C')$$

where $\mathbb{R}(A', C')$ is the reachable subspace that would be associated with the system

$$d(k+1) = A'd(k) + C'e(k)$$

(whose state vector is d and input is e). Reachability and unobservability are said to be **dual** concepts, on account of the preceding connections.

Example 24.1 (Harmonic Oscillator)

Suppose the position and velocity of a particle that is oscillating harmonically at a frequency of ω are sampled with a sampling period of Δ seconds. A state-space description of the sampled state vector is given by:

$$\mathbf{x}(k+1) = \left[\begin{array}{cc} \cos(\omega\Delta) & \frac{1}{\omega}\sin(\omega\Delta) \\ -\omega\sin(\omega\Delta) & \cos(\omega\Delta) \end{array} \right] \mathbf{x}(k),$$

where x_1 is the position of the particle, and x_2 is the velocity. Suppose the measured output is x_1, *i.e.*

$$y(k) = C\mathbf{x}(k) = \left[\begin{array}{cc} 1 & 0 \end{array} \right] \mathbf{x}(k).$$

The subspace of initial conditions that is unobservable over 1 time step is just the nullspace of C:

$$\mathcal{N}(C) = \mathcal{N} \left[\begin{array}{cc} 1 & 0 \end{array} \right] = \mathrm{span}\left\{ \left[\begin{array}{c} 0 \\ 1 \end{array} \right] \right\}.$$

The subspace unobservable over 2 time steps is

$$\mathcal{N} \left[\begin{array}{cc} 1 & 0 \\ \cos(\omega\Delta) & \frac{1}{\omega}\sin\omega\Delta \end{array} \right]$$

$$= \left\{ \begin{array}{ll} \mathrm{span}\left\{ \left[\begin{array}{c} 0 \\ 1 \end{array} \right] \right\} & \text{if } \omega\Delta = N\pi, \ N \in \mathbb{Z} \\ \\ \emptyset & \text{if } \omega\Delta \neq N\pi \end{array} \right.$$

This simply says that if the sampling interval is an integer multiple of one-half the oscillation period, the system will not be observable. Note that an unobservable system with $N = 1$ corresponds to sampling at exactly the Nyquist rate; the system is always observable at sampling frequencies higher than the Nyquist rate.

24.3.1 Modal Interpretation of Unobservability

We start with the time-domain representation of the output for $\mathbf{u}(k) \equiv 0$. If A is diagonalizable, this yields

$$\begin{align} \mathbf{y}(k) &= CA^k\mathbf{x}(0) & (24.6) \\ &= C\sum_{i=1}^{n}\lambda_i^k\mathbf{v}_i\mathbf{w}_i^T\mathbf{x}(0) \\ &= \sum_{i=1}^{n}C\mathbf{v}_i[\mathbf{w}_i^T\mathbf{x}(0)]\lambda_i^k. & (24.7) \end{align}$$

Suppose there exists an eigenvector $\mathbf{v}_i^*$, $1 \leq i^* \leq n$, such that $C\mathbf{v}_{i*} = 0$. Is there an initial state such that $\mathbf{y}(k) \equiv 0$, $\forall\, k \geq 0$? If we choose $\mathbf{x}(0) = \mathbf{v}_{i*}$, then, referring to (24.7), we see that

$$\mathbf{w}_i^T\mathbf{x}(0) = \mathbf{w}_i^T\mathbf{v}_{i*} = 0, \quad \text{for} \quad i \neq i^*.$$

But when $i = i^*$ in (24.7), $C\mathbf{v}_{i*} = 0$. Hence $\mathbf{y}(k) \equiv 0$, $\quad \forall\, k > 0$.

24.3.2   The Observability Gramian

We begin by defining the k-step observability Gramian as

$$
\begin{aligned}
\mathcal{Q}_k &= \mathcal{O}_k^T \mathcal{O}_k \\
&= \sum_{i=0}^{k-1} (A^i)^T C^T C A^i.
\end{aligned}
$$

The unobservable space over k steps is evidently the nullspace of $\mathcal{Q}_k$. The system is observable if and only if $\mathrm{rank}(\mathcal{Q}_n) = n$. If the system is stable, then we can define **the** *observability Gramian* as

$$
\mathcal{Q} = \lim_{k \to \infty} \mathcal{Q}_k = \sum_{i=1}^{\infty} (A^i)^T C^T C A^i.
$$

$\mathcal{Q}$ satisfies a Lyapunov equation that is quite similar to the reachability gramian, *i.e.*,

$$
A^T \mathcal{Q} A - \mathcal{Q} = -C^T C.
$$

24.4   Continuous-Time Analysis

As with reachability, the discussion of observability for continuous-time systems is algebraically very similar to the discussion for discrete-time systems. We begin with a theorem.

Theorem 24.3 For continuous-time systems, the following conditions are equivalent:

1. $\mathbf{x}(0)$ is unobservable in time T.

2. $\mathbf{x}(0)$ is unobservable in any time.

3. $\mathcal{O}_n \mathbf{x}(0) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \mathbf{x}(0) = 0.$

Proof: 1) $\Longrightarrow$ 2):

If ξ is unobservable in time T, then for $\mathbf{x}(0) = \xi$, if $\mathbf{u}(t) = 0$ then $\mathbf{y}(t) = 0, \quad 0 \le t < T$. It follows that

$$
0 = Ce^{At}\xi, \quad 0 \le t < T
$$

From this, it follows that

$$
\begin{aligned}
Ce^{A0}\xi &= C\xi = 0 \\
\frac{d}{dt}Ce^{At}\xi \bigg|_{t=0} &= CA\xi = 0 \\
&\vdots \\
\frac{d^k}{dt^k}Ce^{At}\xi \bigg|_{t=0} &= CA^k\xi = 0.
\end{aligned}
$$

Now, since the power series representation is valid $\forall\, t \geq 0$, this implies that

$$Ce^{At}\xi = 0, \quad \forall\, t \geq 0.$$

Hence the result.

2) $\Longrightarrow$ 1):

This is immediate.

2) $\Longleftrightarrow$ 3): This follows from the Taylor series argument, since by Cayley-Hamilton,

$$\mathcal{N}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = \mathcal{N}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n+k} \end{bmatrix} \quad \forall\, k \geq 0.$$

24.4.1   The Observability Gramian

Define

$$\mathcal{Q}_t = \int_0^t (e^{A\tau})^T C^T C e^{A\tau} d\tau.$$

The system is then observable if and only if $\mathrm{rank}(\mathcal{Q}_t) = n, \quad \forall\, t > 0$. If A is stable, then we may define the observability Gramian as

$$\mathcal{Q} = \int_0^\infty (e^{A\tau})^T C^T C e^{A\tau} d\tau.$$

Once again, $\mathcal{Q}$ satisfies a Lyapunov equation:

$$A^T \mathcal{Q} + \mathcal{Q}A = -C^T C.$$

24.5   Further Results

In view of duality, we can use our reachability results to immediately derive various conclusions, tests, standard and canonical forms, etc., for observable and unobservable systems. We shall simply list the main results:

Fact 1: The unobservable subspace is A-invariant (and is in fact the largest A-invariant subspace contained in the nullspace of C).

Fact 2: An unobservable pair (C, A) can be taken by a similarity transformation to the form

$$A \quad \rightarrow \quad \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix} \tag{24.8}$$

$$C \quad \rightarrow \quad \begin{bmatrix} 0 & C_2 \end{bmatrix} \tag{24.9}$$

where (C_2, A_2) is observable. The unobservable subspace of the system associated with (24.8), (24.9) is given by states of the form $\begin{bmatrix} * \\ 0 \end{bmatrix}$. The eigenvalues of A_1 are the unobservable eigenvalues of the system, while those of A_2 are the observable eigenvalues.

Fact 3: The system is unobservable iff $Cv = 0$ for some right eigenvector v of A, associated with an eigenvalue λ; or equivalently iff

$$\begin{bmatrix} sI - A \\ C \end{bmatrix} \tag{24.10}$$

drops rank for some $s = \lambda$. This λ is an unobservable eigenvalue of the system.

Fact 4: The dual of the problem of controllability to the origin is referred to as *constructability* of the *final* state, *i.e.* instead of trying to uniquely determine the initial state $x(0)$ from input/output measurements over an interval, we wish to determine the final state. In CT, the condition for this reduces to the observability condition, but in DT it turns out that $x(k)$ (for $k \geq n$) can be determined from $u(i)$, $y(i)$ over $[0, k-1]$ iff

$$\mathcal{N}(\mathcal{O}_n) \subset \mathcal{N}(A^n). \tag{24.11}$$

We leave you to prove this. Note that for invertible A we recover the observability condition.

Fact 5: Any single-output (SO) observable n^{th}-order system is similarity transformable to an observability canonical form (dual to the reachability/controllability canonical form presented last lecture) or to an observer canonical form (dual to the controller canonical form of last lecture).

24.5.1 Standard Form for Unobservable Systems

Given an arbitrary system, we can construct a rectangular matrix $T_{\overline{o}}^{n \times r}$, whose columns span the nullspace of the observability matrix $\mathcal{O}_n$. We may then construct T' by selecting $(n - r)$ linearly independent vectors, such that

$$\text{rank}(T^{n \times n}) = \text{rank}\begin{bmatrix} T_{\overline{o}} & T' \end{bmatrix} = n.$$

Since T is invertible, we can perform a similarity transform to generate an equivalent system, where we have:

$$AT = A\begin{bmatrix} T_{\overline{o}} & T' \end{bmatrix} \quad = \quad T\overline{A} = \begin{bmatrix} T_{\overline{o}} & T' \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \tag{24.12}$$

$$CT = C\begin{bmatrix} T_{\overline{o}} & T' \end{bmatrix} \quad = \quad \overline{C} = \begin{bmatrix} 0 & C_1 \end{bmatrix} \tag{24.13}$$

The presence of the zero blocks in the transformed system and output matrices follows from an argument similar to that used for the reachable canonical form coupled with the fact the the unobservable space is also A-invariant. It follows from (24.12) and (24.13) that

$$\overline{\mathcal{O}}_n = \begin{bmatrix} CT \\ CAT \\ \vdots \\ CA^{n-1}T \end{bmatrix} = \begin{bmatrix} 0 & C_1 \\ 0 & C_1 A_3 \\ \vdots & \vdots \\ 0 & C_1 A_3^{n-1} \end{bmatrix}.$$

Now since the transformed system is equivalent to the original one,

$$\text{rank}(\mathcal{O}_n) = \text{rank}(\overline{\mathcal{O}}_n).$$

Theorem 24.4 (Modal Test) A continuous-time system is observable if and only if

$$\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n, \quad \forall \lambda \in \mathbb{C}$$

Proof: The proof follows from the observable canonical form.

Exercises

Exercise 24.1 (a) Given the *observable* nth-order DT system $x(k + 1) = Ax(k) + Bu(k)$, $y(k) = Cx(k) + Du(k)$, show that we can uniquely determine the initial condition $x(0)$ *from output measurements alone*, i.e. without knowledge of the inputs $u(i)$, if $D = 0$, $CB = 0$, $CAB = 0, \ldots, CA^{n-2}B = 0$.

(b) (Optional) Prove that the sufficient condition in (a) is also *necessary* in the case where the output y is scalar.

(c) Verify in the case of a single-input, single-output (SISO) system that the condition in (a) corresponds to the transfer function of the system having no (finite) zeros, only poles.

Exercise 24.2 Consider the system $x(k+1) = Ax(k)$, $y(k) = Cx(k)$, and suppose that A and C have the following form:

$$
A = \begin{pmatrix}
A_{11} & A_{12} & 0 & \ldots & 0 \\
A_{21} & A_{22} & A_{23} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{k-1,1} & A_{k-1,2} & A_{k-1,3} & \ldots & A_{k-1,k} \\
A_{k1} & A_{k2} & A_{k3} & \ldots & A_{kk}
\end{pmatrix}
$$

$$
C = (\, C_1 \quad 0 \quad 0 \quad \ldots \quad 0 \,)
$$

with the $A_{i-1,i}$ and C_1 all having full column rank, and with the A_{ii} all square.

(a) Show that the system is observable. (Hint: Show first that if P and Q have full column rank, then PQ has full column rank.)

(b) What can you say about the system if $A_{k-1,k}$, instead of having full column rank, is actually 0?

[The results of this problem form the basis for one of the best numerical methods for checking observability (or, using a dual set of results and computations, for checking reachability). The point is that orthogonal (and therefore numerically well behaved) similarity transformations can be used to bring a (C, A) pair to the structured form above.]

Exercise 24.3 (a) Consider the CT single-output nth-order observable system $\dot{x}(t) = Ax(t)$, $y(t) = cx(t)$, where c is an n-component vector. Suppose we observe only *samples* of the output, taken at times $t = kT$ for integer k and some fixed T. Write down a DT state-space model that describes the evolution of these output samples.

(b) Find a necessary and sufficient condition, stated in terms of the eigenvalues of A, for your DT model in (a) to be observable. (Hint: First show that there is no loss of generality in taking A to be in Jordan form.)

Exercise 24.4 Consider an observable single output LTI system. Find the smallest perturbation of the C row vector, $\|\delta\|_2$, so that the system becomes unobservable.

Exercise 24.5 Consider a SISO, stable, discrete-time system with a state-space description given by $(A, B, C, 0)$, both reachable and observable. The dimension of the state vector is equal to n. We are interested in studying the effects of past inputs on future outputs in a precise fashion. It turns out that this information is quite valuable in model reduction problems, but we will not look into that here. Recall that the energy of a signal u is given by

$$\|u\|_2 = (u^T u)^{\frac{1}{2}} = (\sum_{t=-\infty}^{\infty} u(t)^2)^{\frac{1}{2}}$$

(Don't be afraid to transpose infinite vectors or matrices).

The precise quantity that we would like to compute is:

$$\alpha = \max_u \{\sum_{t=0}^{\infty} y(t)^2 \mid \sum_{t=-\infty}^{-1} u(t)^2 \leq 1, \ u(k) = 0 \ \forall k \geq 0\}$$

In words, we want to find the maximal achievable energy of the output after $t = 0$ which is the response to an input starting at $-\infty$ and ending at -1, i.e., the has the following form

$$u = (\ldots \quad u(-N) \quad u(-N+1) \quad \ldots \quad u(-1) \quad 0 \quad \ldots)^T$$

with energy less than or equal to one.

(a) Find an expression for $x(0)$ in terms of the above input. Does the value $x(-\infty)$ enter your expression. Explain.

(b) Can any $\xi \in \mathbb{R}^n$ be achieved by some choice of an input of the above form? If so, find an expression of the minimum energy input, u_{min} that achieves the value $x(0) = \xi$. Compute the square of the energy of u_{min}. Write this expression in terms of the Reachability Gramian, and denote it by $\alpha_1(\xi)$.

(c) If some input u_{min} results in $x(0) = \xi$, write an expression of the output for $t \geq 0$. Compute the square of the energy of the output (for $t \geq 0$) as a function of ξ. Write this in an expression involving the Observability Gramian and denote it by $\alpha_2(\xi)$.

(d) Argue that α is given by
$$\alpha = \max_{\xi} \{\alpha_2(\xi) \mid \alpha_1(\xi) \leq 1\}$$

.

(e) Prove that
$$\alpha = \lambda_{max}\{\mathcal{Q}\mathcal{P}\}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are the reachability and observability gramians of the system, and λ_{max} denotes the maximum eigenvalue of a matrix.

(Hint: Use the fact that any symmetric positive semi-definite matrix can be written as $M^T M$. Also you may need the fact that $\lambda_{max}(M^T G M) = \lambda_{max}(G M M^T)$).

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