Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

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Lecture 2: Differential Equations As System Models[1]

Ordinary differential requations (ODE) are the most frequently used tool for modeling continuous-time nonlinear dynamical systems. This section presens results on existence of solutions for ODE models, which, in a systems context, translate into ways of proving well-posedness of interconnections.

2.1 ODE models and their solutions

Ordinary differential equations are used to describe responses of a dynamical system to all possible inputs and initial conditions. Equations which do not have a solution for some valid inputs and initial conditions do not define system's behavior completely, and, hence, are inappropriate for use in analysis and design. This is the reason a special attention is paid in this lecture to the general question of existence of solution of differential equation.

2.1.1 ODE and their solutions

An *ordinary differential equation* on a subset $Z \subset \mathbf{R}^n \times \mathbf{R}$ is defined by a function $a: Z \mapsto \mathbf{R}^n$. Let T be a non-empty convex subset of $\mathbf{R}$ (i.e. T can be a single point set, or an open, closed, or semi-open interval in $\mathbf{R}$). A function $x: T \mapsto \mathbf{R}^n$ is called a *solution* of the ODE

$$\dot{x}(t) = a(x(t), t) \tag{2.1}$$

if $(x(t), t) \in Z$ for all $t \in T$, and

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} a(x(t), t)dt \quad \forall\ t_1, t_2 \in T. \tag{2.2}$$

[1] Version of September 10, 2003

The variable t is usually referred to as the "time".

Note the use of an integral form in the formal definition (2.2): it assumes that the function $t \mapsto a(x(t), t)$ is integrable on T, but does not require $x = x(t)$ to be differentiable at any particular point, which turns out to be convenient for working with discontinuous input signals, such as steps, rectangular impulses, etc.

Example 2.1 Let sgn denote the "sign" function $\text{sgn}: \mathbf{R} \to \{0, -1, 1\}$ defined by

$$\text{sgn}(y) = \begin{cases} 1, & y > 0, \\ 0, & y = 0, \\ -1, & y < 0. \end{cases}$$

The notation

$$\dot{x} = -\text{sgn}(x), \tag{2.3}$$

which can be thought of as representing the action of an on/off negative feedback (or describing behavior of velocity subject to dry friction), refers to a differential equation defined as above with $n = 1$, $Z = \mathbf{R} \times \mathbf{R}$ (since $\text{sgn}(x)$ is defined for all real x, and no restrictions on x or the time variable are explicitly imposed in (2.3)), and $a(x, t) = \text{sgn}(x)$. It can be verified[2] that all solutions of (2.3) have the form

$$x(t) = \max\{c - t, 0\} \quad \text{or} \quad x(t) = \min\{t - c, 0\},$$

where c is an arbitrary real constant. These solutions are not differentiable at the critical "stopping moment" $t = c$.

2.1.2 Standard ODE system models

Ordinary differential equations can be used in many ways for modeling of dynamical systems. The notion of a *standard* ODE system model describes the most straightforward way of doing this.

Definition A *standard* ODE model $\mathcal{B} = \text{ODE}(f, g)$ of a system with input $v = v(t) \in V \subset \mathbf{R}^m$ and output $w(t) \in W \subset \mathbf{R}^k$ is defined by a subset $X \subset \mathbf{R}^n$, two functions $f: X \times V \times \mathbf{R}_+ \mapsto \mathbf{R}^n$ and $g: X \times V \times \mathbf{R}_+ \mapsto W$, and a subset $X_0 \subset X$, so that the behavior set $\mathcal{B}$ of the system consists of all pairs (v, w) of signals such that $v(t) \in V$ for all t, and there exist a solution $x: \mathbf{R}_+ \mapsto X$ of the differential equation

$$\dot{x}(t) = f(x(t), v(t), t) \tag{2.4}$$

such that $x(0) \in X_0$ and

$$w(t) = g(x(t), v(t), t). \tag{2.5}$$

A special case of this definition, when the input v is not present, defines an *autonomous* system.

[2] Do it as an excercise!

2.1.3 Well-posedness of standard ODE system models

As it was mentioned before, not all ODE models are adequate for design and analysis purposes. The notion of *well-posedness* introduces some typical constraints aimed at insuring their applicability.

Definition A standard ODE model ODE(f, g) is called *well posed* if for every signal $v(t) \in V$ and for every solution $x_1 : [0, t_1] \mapsto X$ of (2.4) with $x_1(0) \in X_0$ there exists a solution $x : \mathbf{R}_+ \mapsto X$ of (2.4) such that $x(t) = x_1(t)$ for all $t \in [0, t_1]$.

The ODE from Example 2.1.1 can be used to define a standard autonomous ODE system model

$$\dot{x}(t) = -\text{sgn}(x(t)), \quad w(t) = x(t),$$

where $V = X = X_0 = \mathbf{R}$, $f(x, v, t) = -\text{sgn}(x)$ and $g(x, v, t) = x$. It can be verified that this autonomous system is well-posed. However, introducing an input into the model destroys well-posedness, as shown in the following example.

Example 2.2 Consider the standard ODE model

$$\dot{x}(t) = -\text{sgn}(x(t)) + v(t), \quad w(t) = x(t), \tag{2.6}$$

where $v(t)$ is an unconstrained scalar input. Here

$$V = X = X_0 = \mathbf{R}, \quad f(x, v, t) = -\text{sgn}(x) + v, \quad g(x, v, t) = x.$$

While this model appears to describe a physically plausible situation (velocity dynamics subject to dry friction and external force input v), the model is not well-posed.

To *prove* this, consider the input $v(t) = 0.5 = \text{const}$. It is sufficient to show that no solution of the ODE

$$\dot{x}(t) = 0.5 - \text{sgn}(x(t))$$

satisfying $x(0) = 0$ exists on a time interval $[0, t_f]$ for $t_f > 0$. Indeed, let $x = x(t)$ be such solution. As an integral of a bounded function, $x = x(t)$ witll be a continuous function of time. A continuous function over a compact interval always achieves a maximum. Let $t_m \in [0, t_f]$ be an argument of the maximum over $t \in [0, t_f]$.

If $x(t_m) > 0$ then $t_m > 0$ and, by continuity, $x(t) > 0$ in a neighborhood of t_m, hence there exists $\epsilon > 0$ such that $x(t) > 0$ for all $t \in [t_m - \epsilon, t_m]$. According to the differential equation, this means that $x(t_m - \epsilon) = x(t_m) + 0.5\epsilon > x(t_m)$, which contradicts the selection of t_m as an argument of maximum. Hence $\max x(t) = 0$. Similarly, $\min x(t) = 0$. Hence $x(t) = 0$ for all t. But the constant zero function does not satisfy the differentlial equation. Hence, no solution exists.

It can be shown that the absense of solutions in Example 2.1.3 is caused by lack of continuity of function $f = f(x, v, t)$ with respect to x (discontinuity with respect to v and t would not cause as much trouble).

2.2 Existence of solutions for continuous ODE

This section contains fundamental results establishing existence of solutions of differential equations with a continuous right side.

2.2.1 Local existence of solutions for continuous ODE

In this subsection we study solutions $x: \ [t_0, t_f] \mapsto \mathbf{R}^n$ of the standard ODE

$$\dot{x}(t) = a(x(t), t) \tag{2.7}$$

(same as (2.1)), subject to a given *initial condition*

$$x(t_0) = x_0. \tag{2.8}$$

Here $a: \ Z \mapsto \mathbf{R}^n$ is a given continuous function, defined on $Z \subset \mathbf{R}^n \times \mathbf{R}$. It turns out that a solution $x = x(t)$ of (2.7) with initial condition (2.8) exists, at least on a sufficiently short time interval, whenever the point $z_0 = (x_0, t_0)$ lies, in a certain sense, in the *interior* of Z.

Theorem 2.1 *Assume that for some $r > 0$*

$$D_r(x_0, t_0) = \{(\bar{x}, t) \in \mathbf{R}^n \times \mathbf{R} : \ |\bar{x} - x_0| \leq r, \ t \in [t_0, t_0 + r]\}$$

is a subset of Z. Let

$$M = \max\{|a(\bar{x}, t)| : \ (\bar{x}, t) \in D_r(x_0, t_0)\}.$$

Then, for

$$t_f = \min\{t_0 + r/M, t_0 + r\},$$

there exists a solution $x: \ [t_0, t_f] \mapsto \mathbf{R}^n$ of (2.7) satisfying (2.8). Moreover, any such solution also satisfies $|x(t) - x_0| \leq r$ for all $t \in [t_0, t_f]$.

Example 2.3 The ODE

$$\dot{x}(t) = c_0 + c_1 \cos(t) + x(t)^2,$$

where c_0, c_1 are given constants, belongs to the class of *Riccati equations*, which play a prominent role in the linear system theory. According to Theorem 2.1, for any initial condition $x(0) = x_0$ there exists a solution of the Riccati equation, defined on some time interval $[0, t_f]$ of positive length. This does not mean, however, that the corresponding autonomous system model (producing output $w(t) = x(t)$) is well-posed, since such solutions are not necessarily extendable to the complete time half-line $[0, \infty)$.

2.2.2 Maximal solutions

If $x_1 : [t_0, t_1] \mapsto \mathbf{R}^n$ and $x_2 : [t_1, t_2] \mapsto \mathbf{R}^n$ are both solutions of (2.7), and $x_1(t_1) = x_2(t_1)$, then the function $x : [t_0, t_2] \mapsto \mathbf{R}^n$, defined by

$$x(t) = \begin{cases} x_1(t), & t \in [t_0, t_1], \\ x_2(t), & t \in [t_1, t_2], \end{cases}$$

(i.e. the result of concatenating x_1 and x_2) is also a solution of (2.7). This means that some solutions of (2.7) can be *extended* to a larger time interval.

A solution $x : T \mapsto \mathbf{R}^n$ of (2.7) is called *maximal* if there exists no other solution $\bar{x} : \bar{T} \mapsto \mathbf{R}^n$ for which T is a proper subset of $\bar{T}$, and $\bar{x}(t) = x(t)$ for all $t \in T$. In particular, well-posedness of standard ODE system models contains the requirement that all maximal solutions must be defined on the whole time-line $t \in [0, \infty)$.

The following theorem gives a useful characterization of maximal solutions.

Theorem 2.2 *Let X be an open subset of $\mathbf{R}^n$. Let $a : X \times \mathbf{R} \mapsto \mathbf{R}^n$ be a continuous function. Then all maximal solutions of (2.7) are defined on open intervals and, whenever such solution $x : (t_0, t_1) \mapsto X$ has a finite interval end $\bar{t} = t_0 \in \mathbf{R}$ or $\bar{t} = t_1 \in \mathbf{R}$ (as opposed to $t_0 = -\infty$ or $t_1 = \infty$), there exists no sequence $t_k \in (t_0, t_1)$ such that t_k converges to $\bar{t}$ while $x(t_k)$ converges to a limit in X.*

In other words, in the absense of a-priori constraints on the time variable, a solution is not extendable only if $x(t)$ converges to the boundary of the set on which a is defined. In the most typical situation, the domain Z of f in (2.4) is $\mathbf{R}^n \times \mathbf{R}_+$, which means no a-priori constraints on either x or t. In this case, according to Theorem 2.2, a solution $x = x(t)$ not extendable over a finite time interval $[0, t_f)$, $t_f < \infty$, must satisfy the condition

$$\lim_{t \to t_f} |x(t)| = \infty.$$

In Example 2.2.1 with $c_0 = 1$, $c_1 = 0$, one maximal ODE solution is $x(t) = \tan(t)$, defined for $t \in (-\pi/2, \pi/2)$. It cannot be extended on either side because $|x(t)| \to \infty$ as $t \to \pi/2$ or $t \to -\pi/2$.

2.2.3 Discontinuous dependence on time

The ODE describing systems dynamics are frequently discontinuous with respect to the time variable. Indeed, the standard ODE system model includes

$$\dot{x}(t) = f(x(t), v(t), t),$$

where $v = v(t)$ is an input, and the ODE becomes discontinuous with respect to t whenever v is a rectangular impulse etc. As long as the time instances at which $a(x, t)$ is discontinuous for a fixed *finite* set $t_1 < t_2 < \cdots < t_n$, Theorem 2.1 can be applied separately to the time intervals $[t_{k-1}, t_k]$. However, when the location of discontinuities depends on x, or when they cannot be counted in an increasing order, a stronger result is needed. It turns out that the dependence on time needs only be integrable, as long as dependence on x is continuous.

Theorem 2.3 *Assume that for some $r > 0$*

(a) the set
$$D_r(x_0, t_0) = \{(x,t) \in \mathbf{R}^n \times \mathbf{R} : \ |x - x_0| \le r, \ t \in [t_0, t_0 + r]\}$$
is a subset of Z;

(b) the function $t \mapsto a(x(t), t)$ is integrable on $[t_0, t_0 + r]$ for every continuous function $x : \ [t_0, t_0 + r] \mapsto \mathbf{R}^n$ satisfying $|x(t) - x_0| \le r$ for all $t \in [t_0, t_0 + r]$;

(c) for every $\epsilon > 0$ there exists $\delta > 0$ such that
$$\int_{t_0}^{t_0+r} |a(x_1(t), t) - a(x_2(t), t)| dt < \epsilon$$
whenever $x_1, x_2 : \ [t_0, t_0 + r] \mapsto \mathbf{R}^n$ are continuous functions satisfying $|x_k(t) - x_0| \le r$ and $|x_1(t) - x_2(t)| < \delta$ for all $t \in [t_0, t_0 + r]$.

Then, for some $t_f \in (t_0, t_0 + r)$ there exists a solution $x : \ [t_0, t_f] \mapsto \mathbf{R}^n$ of (2.7) satisfying (2.8).

Example 2.4 Theorem 2.3 can be used to show that the differential equation
$$\dot{x}(t) = \begin{cases} t^{-1/3} x(t), & t > 0 \\ 0, & t = 0, \end{cases} \qquad x(0) = x_0$$
does have a solution on $[0, \infty)$ for every $x_0 \in \mathbf{R}$ (in this particular case the solutions can be found analytically). Indeed, for every continuous function $x : \ [0, \infty) \mapsto \mathbf{R}$ the function $t \mapsto t^{-1/3} x(t)$ for $t > 0$ is integrable over every finite interval, and the inequality
$$\int_0^{t_1} |t^{-1/3} x_1(t) - t^{-1/3} x_2(t)| dt \le \int_0^{t_1} t^{-1/3} dt \max_{t \in [0, t_1]} |x_1(t) - x_2(t)|$$
holds.

On the contrary, the differential equation
$$\dot{x}(t) = \begin{cases} t^{-1} x(t), & t > 0 \\ 0, & t = 0, \end{cases} \qquad x(0) = x_0$$
does not have a solution on $[0, \infty)$ for every $x_0 \neq 0$. Indeed, if $x : \ [0, t_1] \mapsto \mathbf{R}$ is a solution for some $t_1 > 0$ then
$$\frac{d}{dt}\left(\frac{x(t)}{t}\right) = 0$$
for all $t \neq 0$. Hence $x(t) = ct$ for some constant c, and $x(0) = 0$.

2.2.4 Differential inclusions

Let X be a subset of $\mathbf{R}^n$, and let $\eta: X \to 2^{\mathbf{R}^n}$ be a function which maps every point of X to a *subset* of $\mathbf{R}^n$. Such a function defines a *differential inclusion*

$$\dot{x}(t) \in \eta(x(t)). \tag{2.9}$$

By a solution of (2.1) on a convex subset T of $\mathbf{R}$ we mean a function $x: T \mapsto X$ such that

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} u(t)dt \quad \forall\ t_1, t_2 \in T$$

for some integrable function $u: T \mapsto \mathbf{R}^n$ satisfying the inclusion $u(t) \in \eta(x(t))$ for all $t \in T$. It turns out that differential inclusions are a convenient, though not always adequate, way of re-defining discontinuous ODE to guarantee existence of solutions.

It turns out that differential inclusion (2.9) subject to fixed initial condition $x(t_0) = x_0$ has a solution on a sufficiently small interval $T = [t_0, t_1]$ whenever the set-valued function η is *compact convex set-valued* and *semicontinuous* with respect to its argument (plus, as usually, x_0 must be an interior point of X).

Theorem 2.4 *Assume that for some $r > 0$*

(a) the set

$$B_r(x_0) = \{x \in \mathbf{R}^n : |x - x_0| \le r\}$$

is a subset of X;

(b) for every $\bar{x} \in B_r(x_0)$ the set $\eta(\bar{x})$ is convex;

(c) for every sequence of $\bar{x}_k \in B_r(x_0)$ converging to a limit $\bar{x} \in B_r(x_0)$ and for every sequence $\bar{u}_k \in \eta(\bar{x}_k)$ there exists a subsequence $k = k(q) \to \infty$ as $q \to \infty$ such that the subsequence $\bar{u}_{k(q)}$ has a limit in $\eta(\bar{x})$.

Then the supremum

$$M = \sup\{|\bar{u}| : \ \bar{u} \in \eta(\bar{x}),\ \bar{x} \in D_r(x_0, t_0)\}$$

is finite, and, for

$$t_f = \min\{t_0 + r/M, t_0 + r\},$$

there exists a solution $x: [t_0, t_f] \mapsto \mathbf{R}^n$ of (2.9) satisfying $x(t_0) = x_0$. Moreover, any such solution also satisfies $|x(t) - x_0| \le r$ for all $t \in [t_0, t_f]$.

The discontinuous differential equation

$$\dot{x}(t) = -\text{sgn}(x(t)) + c,$$

where c is a fixed constant, can be re-defined as a continuous differential inclusion (2.9) by introducing

$$\eta(y) = \begin{cases} \{c-1\}, & y > 0, \\ [c-1, c+1], & y = 0, \\ \{c+1\}, & y < 0. \end{cases}$$

The newly obtained differential inclusion has the "existence of solutions" property, and appears to be compatible with the "dry friction" interpretation of the sign nonlinearity. In particular, with the initial condition $x(0) = 0$, the equation has solutions for every value of $c \in \mathbf{R}$. If $c \in [-1, 1]$, the unique maximal solution is $x(t) \equiv 0$, which corresponds to the friction force "adapting" itself to equalize the external force, as long as it is not too large.

The differential inclusion model is not as compatible with the "on/off controller" interpretation of the sign nonlinearity. In this case, due to the unmodeled feedback loop delays, one expects some "chattering" solutions oscillating rapidly around the point $x_0 = 0$. It is possible to say that, in this particular case, the solutions of (2.9) describe the *limit behavior* of the closed loop solutions as the loop delay approaches zero.