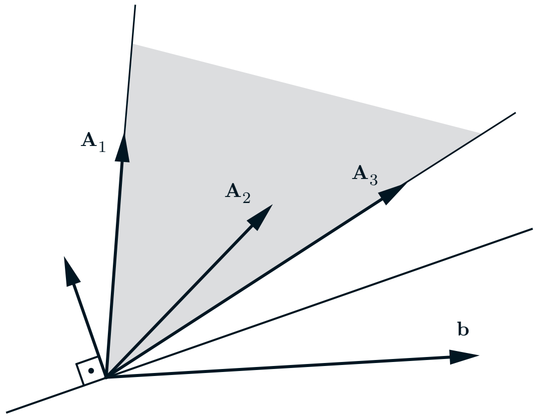
15.081J/6.251J Introduction to Mathematical Programming

Lecture 10: Duality Theory III



1 Outline

SLIDE 1

- Farkas lemma
- Asset pricing
- Cones and extreme rays
- Representation of Polyhedra

2 Farkas lemma

SLIDE 2

Theorem:
Exactly one of the following two alternatives hold:

1. $\exists \boldsymbol{x} \geq \mathbf{0}$ s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.
2. $\exists \boldsymbol{p}$ s.t. $\boldsymbol{p}'\boldsymbol{A} \geq \mathbf{0}'$ and $\boldsymbol{p}'\boldsymbol{b} < 0$.

2.1 Proof

SLIDE 3

"$\Rightarrow$" If $\exists \boldsymbol{x} \geq \mathbf{0}$ s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, and if $\boldsymbol{p}'\boldsymbol{A} \geq \mathbf{0}'$, then $\boldsymbol{p}'\boldsymbol{b} = \boldsymbol{p}'\boldsymbol{A}\boldsymbol{x} \geq 0$
"$\Leftarrow$" Assume there is no $\boldsymbol{x} \geq \mathbf{0}$ s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

$$\begin{array}{lll} (P) \max & \mathbf{0}'\boldsymbol{x} & \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} & \\ & \boldsymbol{x} \geq \mathbf{0} & \end{array} \qquad \begin{array}{ll} (D) \min & \boldsymbol{p}'\boldsymbol{b} \\ \text{s.t.} & \boldsymbol{p}'\boldsymbol{A} \geq \mathbf{0}' \end{array}$$

(P) infeasible $\Rightarrow$ (D) either unbounded or infeasible
Since $\boldsymbol{p} = \mathbf{0}$ is feasible $\Rightarrow$ (D) unbounded
$\Rightarrow \exists \boldsymbol{p}: \quad \boldsymbol{p}'\boldsymbol{A} \geq \mathbf{0}'$ and $\boldsymbol{p}'\boldsymbol{b} < 0$

3 Asset Pricing

SLIDE 4

- n different assets
- m possible states of nature
- one dollar invested in some asset i, and state of nature is s, we receive a payoff of r_{si}
- $m \times n$ payoff matrix:

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & \dots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{m1} & \dots & r_{mn} \end{bmatrix}$$

SLIDE 5

- x_i: amount held of asset i. A portfolio of assets is $\boldsymbol{x} = \big(x_1, \dots, x_n\big)$.
- A negative value of x_i indicates a "short" position in asset i: this amounts to selling $|x_i|$ units of asset i at the beginning of the period, with a promise to buy them back at the end. Hence, one must pay out $r_{si}|x_i|$ if state s occurs, which is the same as receiving a payoff of $r_{si}x_i$

SLIDE 6

- Wealth in state s from a portfolio $\boldsymbol{x}$

$$w_s = \sum_{i=1}^{n} r_{si} x_i.$$

- $\boldsymbol{w} = \big(w_1, \dots, w_m\big)$, $\boldsymbol{w} = \boldsymbol{R}\boldsymbol{x}$
- p_i: price of asset i, $\boldsymbol{p} = \big(p_1, \dots, p_n\big)$
- Cost of acquiring $\boldsymbol{x}$ is $\boldsymbol{p}'\boldsymbol{x}$.

3.1 Arbitrage

SLIDE 7

- Central problem: Determine p_i
- **Absence of arbitrage:** no investor can get a guaranteed nonnegative payoff out of a negative investment. In other words, any portfolio that pays off nonnegative amounts in every state of nature, must have nonnegative cost.

$$\text{if} \quad \boldsymbol{R}\boldsymbol{x} \geq \boldsymbol{0}, \quad \text{then} \quad \boldsymbol{p}'\boldsymbol{x} \geq 0.$$

SLIDE 8

- Theorem: The absence of arbitrage condition holds if and only if there exists a nonnegative vector $\boldsymbol{q} = (q_1, \ldots, q_m)$, such that the price of each asset i is given by
$$p_i = \sum_{s=1}^{m} q_s r_{si}.$$
- Applications to options pricing

4 Cones and extreme rays

4.1 Definitions

SLIDE 9

- A set $C \subset \Re^n$ is a **cone** if $\lambda \boldsymbol{x} \in C$ for all $\lambda \geq 0$ and all $\boldsymbol{x} \in C$
- A polyhedron of the form $P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{Ax} \geq \boldsymbol{0}\}$ is called a polyhedral cone

4.2 Applications

SLIDE 10

- $P = \big\{\boldsymbol{x} \in \Re^n \mid \boldsymbol{Ax} \geq \boldsymbol{b}\big\}$, $\boldsymbol{y} \in P$
- The recession cone at $\boldsymbol{y}$
$$RC = \big\{\boldsymbol{d} \in \Re^n \mid \boldsymbol{y} + \lambda \boldsymbol{d} \in P, \forall\ \lambda \geq 0\big\}$$
- It turns out that
$$RC = \big\{\boldsymbol{d} \in \Re^n \mid \boldsymbol{Ad} \geq \boldsymbol{0}\big\}$$
- RC independent of $\boldsymbol{y}$

SLIDE 11

4.3 Extreme rays

SLIDE 12

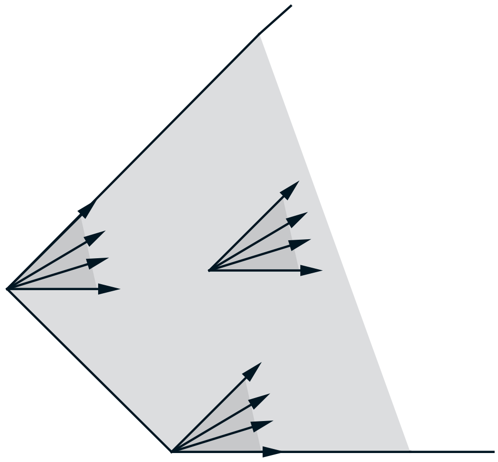
A $\boldsymbol{x} \neq \boldsymbol{0}$ of a polyhedral cone $C \subset \Re^n$ is called an **extreme ray** if there are $n-1$ linearly independent constraints that are active at $\boldsymbol{x}$

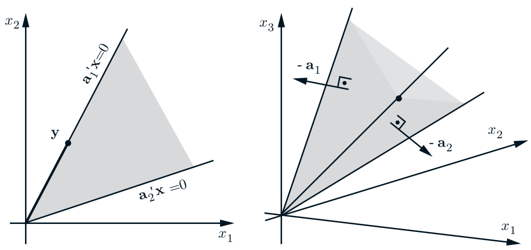
4.4 Unbounded LPs

SLIDE 13

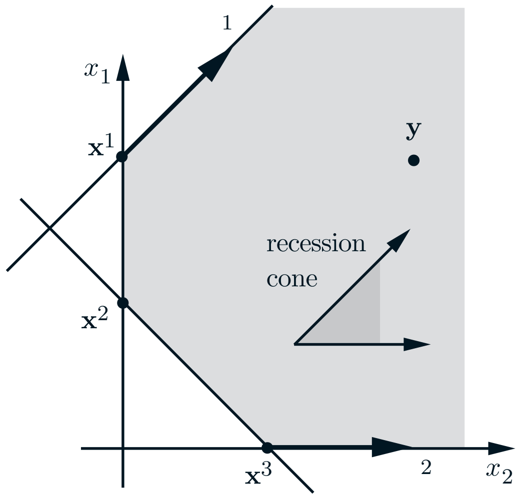
Theorem: Consider the problem of minimizing $\boldsymbol{c}'\boldsymbol{x}$ over a polyhedral cone $C = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}_i'\boldsymbol{x} \geq 0,\ i = 1, \ldots, m\}$ that has zero as an extreme point. The optimal cost is equal to $-\infty$ if and only if some extreme ray $\boldsymbol{d}$ of C satisfies $\boldsymbol{c}'\boldsymbol{d} < 0$. Theorem: Consider the problem of minimizing $\boldsymbol{c}'\boldsymbol{x}$ subject to $\boldsymbol{Ax} \geq \boldsymbol{b}$, and assume that the feasible set has at least one extreme point. The optimal cost is equal to $-\infty$ if and only if some extreme ray $\boldsymbol{d}$ of the feasible set satisfies $\boldsymbol{c}'\boldsymbol{d} < 0$. SLIDE 14

What happens when the simplex method detects an unbounded problem?




x_2
$\mathbf{a}_1'\mathbf{x}=0$
$\mathbf{y}$
$\mathbf{a}_2'\mathbf{x}=0$
x_1
x_3
- $\mathbf{a}_1$
- $\mathbf{a}_2$
x_2
x_1

(a) (b)



5 Resolution Theorem

SLIDE 15

$$P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{Ax} \geq \boldsymbol{b}\}$$

be a nonempty polyhedron with at least one extreme point. Let $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^k$ be the extreme points, and let $\boldsymbol{w}^1, \ldots, \boldsymbol{w}^r$ be a complete set of extreme rays of P.

$$Q = \left\{ \sum_{i=1}^{k} \lambda_i \boldsymbol{x}^i + \sum_{j=1}^{r} \theta_j \boldsymbol{w}^j \;\middle|\; \lambda_i \geq 0,\ \theta_j \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

Then, $Q = P$.

5.1 Example

SLIDE 16

$$\begin{aligned} x_1 - x_2 &\geq -2 \\ x_1 + x_2 &\geq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

SLIDE 17

- Extreme points: $\boldsymbol{x}^1 = (0,2)$, $\boldsymbol{x}^2 = (0,1)$, and $\boldsymbol{x}^3 = (1,0)$.
- Extreme rays $\boldsymbol{w}^1 = (1,1)$ and $\boldsymbol{w}^2 = (1,0)$.
- $$\boldsymbol{y} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \boldsymbol{x}^2 + \boldsymbol{w}^1 + \boldsymbol{w}^2.$$

5.2 Proof

SLIDE 18

- $Q \subset P$. Let $\boldsymbol{x} \in Q$:

$$\boldsymbol{x} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}^i + \sum_{j=1}^{r} \theta_j \boldsymbol{w}^j$$

$\lambda_i,\ \theta_j \geq 0\ \sum_{i=1}^{k} \lambda_i = 1.$

- $\boldsymbol{y} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}^i \in P$ and satisfies $\boldsymbol{Ay} \geq \boldsymbol{b}$.
- $\boldsymbol{Aw}^j \geq \boldsymbol{0}$ for every j: $\boldsymbol{z} = \sum_{j=1}^{r} \theta_j \boldsymbol{w}^j$ satisfies $\boldsymbol{Az} \geq \boldsymbol{0}$.
- $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$ satisfies $\boldsymbol{Ax} \geq \boldsymbol{b}$ and belongs to P.

SLIDE 19

For the reverse, assume there is a $\boldsymbol{z} \in P$, such that $z \notin Q$.

$$\begin{array}{ll}
\max & \displaystyle\sum_{i=1}^{k} 0\lambda_i + \sum_{j=1}^{r} 0\theta_j \\
\text{s.t.} & \displaystyle\sum_{i=1}^{k} \lambda_i \boldsymbol{x}^i + \sum_{j=1}^{r} \theta_j \boldsymbol{w}^j = \boldsymbol{z} \\
& \displaystyle\sum_{i=1}^{k} \lambda_i = 1 \\
& \lambda_i \geq 0, \qquad i = 1, \ldots, k, \\
& \theta_j \geq 0, \qquad j = 1, \ldots, r,
\end{array}$$

Is this feasible?

SLIDE 20

- Dual

$$\begin{array}{lll}
\min & \boldsymbol{p}'\boldsymbol{z} + q & \\
\text{s.t.} & \boldsymbol{p}'\boldsymbol{x}^i + q \geq 0, & i = 1, \ldots, k, \\
& \boldsymbol{p}'\boldsymbol{w}^j \geq 0, & j = 1, \ldots, r.
\end{array}$$

- This is unbounded. Why?
- There exists a feasible solution $(\boldsymbol{p}, q)$ whose cost $\boldsymbol{p}'\boldsymbol{z} + q < 0$
- $\boldsymbol{p}'\boldsymbol{z} < \boldsymbol{p}'\boldsymbol{x}^i$ for all i and $\boldsymbol{p}'\boldsymbol{w}^j \geq 0$ for all j.

SLIDE 21

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$$\begin{array}{ll}
\min & \boldsymbol{p}'\boldsymbol{x} \\
\text{s.t.} & \boldsymbol{Ax} \geq \boldsymbol{b}.
\end{array}$$

- If the optimal cost is finite, there exists an extreme point $\boldsymbol{x}^i$ which is optimal. Since $\boldsymbol{z}$ is a feasible solution, we obtain $\boldsymbol{p}'\boldsymbol{x}^i \leq \boldsymbol{p}'\boldsymbol{z}$, which is a contradiction.
- If the optimal cost is $-\infty$, there exists an extreme ray $\boldsymbol{w}^j$ such that $\boldsymbol{p}'\boldsymbol{w}^j < 0$, which is again a contradiction

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