6.253: Convex Analysis and Optimization
Homework 5

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Problem 1

Consider the convex programming problem

$$\begin{aligned} &\underset{x}{\text{minimize}} && f(x) \\ &\text{subject to} && x \in X, \quad g(x) \le 0, \end{aligned}$$

of Section 5.3, and assume that the set X is described by equality and inequality constraints as

$$X = \{x \mid l_i(x) = 0,\ i = 1, \ldots, \bar{m},\ g_j(x) \le 0,\ j = r+1, \ldots, \bar{r}\}.$$

Then the problem can alternatively be described without an abstract set constraint, in terms of all of the constraint functions

$$l_i(x) = 0, \quad i = 1, \ldots, \bar{m}, \qquad g_j(x) \le 0, \quad j = 1, \ldots, \bar{r}.$$

We call this the *extended representation* of (P). Show if there is no duality gap and there exists a dual optimal solution for the extended representation, the same is true for the original problem.

Problem 2

Consider the class of problems

$$\begin{aligned} &\underset{x}{\text{minimize}} && f(x) \\ &\text{subject to} && x \in X, \qquad g_j(x) \le u_j, \quad j = 1, \ldots, r, \end{aligned}$$

where $u = (u_1, \ldots, u_r)$ is a vector parameterizing the right-hand side of the constraints. Given two distinct values $\bar{u}$ and $\tilde{u}$ of u, let $\bar{f}$ and $\tilde{f}$ be the corresponding optimal values, and assume that $\bar{f}$ and $\tilde{f}$ are finite. Assume further that $\bar{\mu}$ and $\tilde{\mu}$ are corresponding dual optimal solutions and that there is no duality gap. Show that

$$\tilde{\mu}'(\tilde{u} - \bar{u}) \le \bar{f} - \tilde{f} \le \bar{\mu}'(\tilde{u} - \bar{u}).$$

Problem 3

Let $g_j : R^n \mapsto R$, $j = 1, \ldots, r$, be convex functions over the nonempty convex subset of R^n. Show that the system

$$g_j(x) < 0, \qquad j = 1, \ldots, r,$$

has no solution within X if and only if there exists a vector $\mu \in R^r$ such that

$$\sum_{j=1}^{r} \mu_j = 1, \qquad \mu \geq 0,$$

$$\mu' g(x) \geq 0, \qquad \forall\, x \in X.$$

Hint: Consider the convex program

$$\begin{aligned} &\underset{x,y}{\text{minimize}} && y \\ &\text{subject to} && x \in X,, \quad y \in R, \qquad g_j(x) \leq y, \quad j = 1, \ldots, r. \end{aligned}$$

Problem 4

Consider the problem

$$\begin{aligned} &\underset{x}{\text{minimize}} && f(x) \\ &\text{subject to} && x \in X, \quad g(x) \leq 0, \end{aligned}$$

where X is a convex set, and f and g_j are convex over X. Assume that the problem has at least one feasible solution. Show that the following are equivalent.

(i) The dual optimal value $q^* = \sup_{\mu \in R^r} q(\mu)$ is finite.

(ii) The primal function p is proper.

(iii) The set

$$M = \{(u, w) \in R^{r+1} \mid \text{there is an } x \in X \text{ such that } g(x) \leq u,\ f(x) \leq w\}$$

does not contain a vertical line.

Problem 5

Consider a proper convex function F of two vectors $x \in R^n$ and $y \in R^m$. For a fixed $(\bar{x}, \bar{y}) \in$ *dom*(F), let $\partial_x F(\bar{x}, \bar{y})$ and $\partial_y F(\bar{x}, \bar{y})$ be the subdifferentials of the functions $F(\cdot, \bar{y})$ and $F(\bar{x}, \cdot)$ at $\bar{x}$ and $\bar{y}$, respectively. (a) Show that

$$\partial F(\bar{x}, \bar{y}) \subset \partial_x F(\bar{x}, \bar{y}) \times \partial_y F(\bar{x}, \bar{y}),$$

and give an example showing that the inclusion may be strict in general. (b) Assume that F has the form

$$F(x, y) = h_1(x) + h_2(y) + h(x, y),$$

where h_1 and h_2 are proper convex functions, and h is convex, real-valued, and differentiable. Show that the formula of part (a) holds with equality.

Problem 6

This exercise shows how a duality gap results in nondifferentiability of the dual function. Consider the problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & x \in X, \quad g(x) \le 0, \end{aligned}$$

and assume that for all $\mu \ge 0$, the infimum of the Lagrangian $L(x, \mu)$ over X is attained by at least one $x_\mu \in X$. Show that if there is a duality gap, then the dual function $q(\mu) = \inf_{x \in X} L(x, \mu)$ is nondifferentiable at every dual optimal solution. *Hint*: If q is differentiable at a dual optimal solution μ^*, by the theory of Section 5.3, we must have $\partial q(\mu^*)/\partial \mu_j \le 0$ and $\mu_j^* \partial q(\mu^*)/\partial \mu_j = 0$ for all j. Use optimality conditions for μ^*, together with any vector x_{μ^*} that minimizes $L(x, \mu^*)$ over X, to show that there is no duality gap.

Problem 7

Consider the problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) = 10x_1 + 3x_2 \\ \text{subject to} \quad & 5x_1 + x_2 \ge 4, x_1, x_2 = 0 \text{ or } 1, \end{aligned}$$

(a) Sketch the set of constraint-cost pairs $\{(4 - 5x_1 - x_2, 10x_1 + 3x_2) | x_1, x_2 = 0 \text{ or } 1\}$.
(b)Describe the corresponding MC/MC framework as per Section 4.2.3.
(c) Solve the problem and its dual, and relate the solutions to your sketch in part (a).