Solutions to Homework 11

6.262 Discrete Stochastic Processes
MIT, Spring 2011

Exercise 1.22:

For each of the following random variables, find the interval (r_-, r_+) over which the moment generating function $g(r)$ exists. Determine in each case whether $g_X(r)$ exists at the end points r_- or r_+. For part c), $g(r)$ has no closed form.

Part a) Let λ, θ, be positive numbers and let X have the density

$$f_X(x) = \frac{1}{2}\lambda \exp(-\lambda x); x \geq 0; \qquad f_X(x) = \frac{1}{2}\theta \exp(\theta x); x < 0$$

Solution: The MGF is calculated as:

$$\begin{aligned}
g_X(r) &= \mathbb{E}[e^{rx}] \\
&= \int_{-\infty}^{\infty} f_X(x)e^{rx}\, dx \\
&= \frac{1}{2}\int_{\infty}^{0} \theta e^{\theta x} e^{rx}\, dx + \frac{1}{2}\int_{0}^{\infty} \lambda e^{-\lambda x} e^{rx}\, dx \\
&= 1/2\theta \frac{e^{x(r+\theta)}}{r+\theta}|_{-\infty}^{0} + 1/2\lambda \frac{e^{x(r-\lambda)}}{r-\lambda}|_{0}^{\infty} \\
&= \frac{\theta}{2(\theta+r)} + \frac{\lambda}{2(\lambda-r)}
\end{aligned}$$

As observed, the first integral is infinite if $r \leq -\theta$ and the second integral is infinite if $r \geq \lambda$. So $g_X(r)$ is defined in the interval $(-\theta, \lambda)$ and boundaries are not included in the region of convergence.

Part b) Let Y be a Gaussian random variable with mean m and variance σ^2.

Solution: $f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(y-m)^2}{2\sigma^2})$ for all y.

$$
\begin{aligned}
g_Y(r) &= \mathbb{E}[e^{ry}] \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(y-m)^2}{2\sigma^2}) e^{rx}\, dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-\frac{(y-(m+r\sigma^2))^2}{2\sigma^2} + r^2\sigma^2/2\right) dx \\
&= \exp(r^2\sigma^2/2) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-(m+r\sigma^2))^2}{2\sigma^2}\right) dx \\
&= \exp(r^2\sigma^2/2)
\end{aligned}
$$

$g_Y(r)$ exists for all $-\infty < r < \infty$. But it does not exist for $r = -\infty$ or $r = \infty$.

Part c Let Z be a non-negative random variable with density

$$f_Z(z) = k(1+z)^{-2} \exp(-\lambda z); \qquad z \geq 0.$$

where $\lambda > 0$ and $k = \left[\int_{z\geq 0} (1+z)^2 \exp(-az) dz\right]^{-1}$. Hint: Do not try to evaluate $g_Z(r)$. Instead, investigate values of r for which the integral is finite and infinite.

Solution:

$$
\begin{aligned}
g_Z(r) &= \mathbb{E}[e^{rz}] \\
&= \int_0^{\infty} e^{rz} k(1+z)^{-2} \exp(-\lambda z)\, dz \\
&= k \int_0^{\infty} (1+z)^{-2} \exp\left((r-\lambda)z\right)\, dz
\end{aligned}
$$

This integral is finite if and only if r is less than λ ($r \leq \lambda$). If r is strictly less than λ ($r < \lambda$), since exponential function decays much faster than any polynomial, the integral will be finite. If $r = \lambda$, the integration is over a function that decays as $z^{-\alpha}$ where $\alpha = 2 > 1$, and it will be finite.

<u>**Exercise 1.24:**</u>

Part a) Assume that the MGF of the random variable X exists (i.e., is finite) in the interval (r_-, r_+), $r_- < r < r_+$ throughout. For any finite constant c, express the moment generating function of $X - c$, i.e., $g_{(X-c)}(r)$, in terms of $g_X(r)$ and show that $g_{(X-c)(r)}$ exists for all r in (r_-, r_+). Explain why $g''_{X-c}(r) \geq 0$.

Solution:

$$\begin{aligned} g_{(X-c)}(r) &= \mathbb{E}[e^{r(X-c)}] \\ &= e^{-rc}\mathbb{E}[e^{rX}] \\ &= e^{-rc}g_X(r) \end{aligned}$$

Since $g_{(X-c)}(r) = e^{-rc}g_X(r)$, if $g_X(r)$ exists and is finite in the interval (r_-, r_+), $g_{(X-c)}(r)$ also exists in this interval.

If we define the random variable $Y = X - c$, then $g_Y(r) = g_{(X-c)}(r)$ and $g''_{X-c}(r)$ is equal to $g''_Y(r) = \mathbb{E}[Y^2 e^{ry}]$ which takes non-negative values.

Part b) Show that $g''_{X-c}(r) = \left[g''_X(r) - 2cg'_X(r) + c^2 g_X(r)\right] e^{-rc}$.

Solution:

$$\begin{aligned} g''_{X-c}(r) &= \frac{d^2\mathbb{E}[e^{r(X-c)}]}{dr^2} \\ &= \mathbb{E}[\frac{d^2 e^{r(X-c)}}{dr^2}] \\ &= \mathbb{E}[(X-c)^2 e^{r(X-c)}] \\ &= e^{-rc}\mathbb{E}[(X^2 - 2cX + c^2)e^{rX}] \\ &= e^{-rc}\left[\mathbb{E}[X^2 e^{rX}] - 2c\mathbb{E}[Xe^{rX}] + c^2\mathbb{E}[e^{rX}]\right] \\ &= e^{-rc}\left[g''_X(r) - 2cg'_X(r) + c^2 g_X(r)\right]. \end{aligned}$$

Part c) Use (a) and (b) to show that $g''_X(r)g_X(r) - [g'_X(r)]^2 \geq 0$, Let $\gamma_X(r) = \ln g_X(r)$ and show that $\gamma''_X(r) \geq 0$. Hint: choose $c = g'_X(r)/g_X(r)$.

Solution: In part a) it was proved that $g''_{X-c}(r) \geq 0$ and in part b) $g''_{(X-c)}(r)$ was calculated. Thus, we know that for any c, $g''_{X-c}(r) = e^{-rc}\left[g''_X(r) - 2cg'_X(r) + c^2 g_X(r)\right] \geq 0$. Since $e^{-rc} \geq 0$, thus, $g''_X(r) - 2cg'_X(r) + c^2 g_X(r) \geq 0$. Now choosing $c = g'_X(r)/g_X(r)$:

$$g''_X(r) - 2\frac{g'_X(r)}{g_X(r)}g'_X(r) + \left[\frac{g'_X(r)}{g_X(r)}\right]^2 g_X(r) \geq 0$$

Which gives the desired result immediately.

Defining $\gamma_X(r) = \ln g_X(r)$, we know that $\gamma'_X(r) = \frac{g'_X(r)}{g_X(r)}$ and $\gamma'_X(r) = \frac{g''_X(r)g_X(r) - [g'_X(r)]^2}{g^2_X(r)}$ which is non-negative due to the result above.

Part d) Assume that X is non-deterministic, i.e., that there is no value of α such that $\Pr\{X = \alpha\} = 1$. Show that the inequality sign " $\geq$ " may be replaced by " $>$ " everywhere in (a),(b) and (c).

Solution: All the inequalities were result of the fact that $\mathbb{E}[(X-c)^2 e^{r(X-c)}] \geq 0$. This is because $e^{r(X-c)} > 0$ for all values of $X > -\infty$ and $(X-c)^2 \geq 0$. Now if there is a positive probability that $(X-c)^2 > 0$ or in other terms if X is not a deterministic random variable that takes the value of c with probability 1, then $(X-c)^2 > 0$ and $g''_{(X-c)}(r) > 0$.

Exercise 7.1:

Consider the simple random walk $\{S_n : n \geq 1\}$ of section 7.1.1 with $S_n = X_1 + X_2 + \cdots + X_n$ and $\Pr\{X_i = 1\} = p$; $\Pr\{X_i = -1\} = 1-p$; assume that $p \leq 1/2$,

Part a) Show that $\Pr\left\{\bigcup_{n\geq 1}\{S_n \geq k\}\right\} = \left[\Pr\left\{\bigcup_{n\geq 1}\{S_n \geq 1\}\right\}\right]^k$ for any positive integer k. Hint: Given that the random walk ever reaches the value 1, consider a new random walk starting at that time and explore the probability that the new walk ever reaches a value 1 greater than its starting point.

Solution: Since X_i is equal to 1 or -1, S_n increases or decreases by value of one at each time. In order for S_n to ever reach the value of k, it should first reach the value of 1.

$$
\begin{aligned}
\Pr\left\{\bigcup_{n\geq 1}\{S_n \geq k\}\right\} &= \Pr\left\{\bigcup_{n\geq 1}\{S_n \geq 1\}\right\}\Pr\left\{\bigcup_{n\geq 1}\{S_n \geq k\} | \bigcup_{n\geq 1}\{S_n \geq 1\}\right\} \\
&= \Pr\left\{\bigcup_{n\geq 1}\{S_n \geq 1\}\right\}\Pr\left\{\bigcup_{n\geq 1}\{S_n \geq k-1\}\right\}
\end{aligned}
$$

Doing the same thing iteratively for $\Pr\left\{\bigcup_{n\geq 1}\{S_n \geq k-1\}\right\}$, we get: $\Pr\left\{\bigcup_{n\geq 1}\{S_n \geq k\}\right\} = \left[\Pr\left\{\bigcup_{n\geq 1}\{S_n \geq 1\}\right\}\right]^k$.

Part b) Find a quadratic equation for $y = \Pr\left\{\bigcup_{n\geq 1}\{S_n \geq 1\}\right\}$. Hint: explore each of the two possibilities immediately after the first trial.

Solution:

$$
\begin{aligned}
y = \Pr\left\{\bigcup_{n\geq 1}\{S_n \geq 1\}\right\} &= \Pr\left\{\bigcup_{n\geq 1}\{S_n \geq 1\}|X_1 = 1\right\}\Pr\{X_1 = 1\} + \Pr\left\{\bigcup_{n\geq 1}\{S_n \geq 1\}|X_1 = -1\right\}\Pr\{X \\
&= \Pr\{X_1 = 1\} + \Pr\left\{\bigcup_{n\geq 1}\{S_n \geq 2\}\right\}\Pr\{X_1 = -1\}
\end{aligned}
$$

Where the last equation is because of the fact that the probability that S_n ever reaches the value of 1 conditional on the $X_1 = -1$ is the same event that S_n should reach value of 2 starting from the beginning. Thus, $y = \Pr\left\{\bigcup_{n\geq 1}\{S_n \geq 1\}\right\} = p + (1-p)\Pr\left\{\bigcup_{n\geq 1}\{S_n \geq 1\}\right\}^2 = p + (1-p)y^2$.

Part c) For $p < 1/2$, show that the two roots of this quadratic equation are $p/(1-p)$ and 1. Argue that $\Pr\left\{\bigcup_{n\geq 1}\{S_n \geq 1\}\right\}$ cannot be 1 and thus must be $p/(1-p)$.

Solution: We see that both the values of $p/(1-p)$ and 1 satisfy the equation $y = p+(1-p)y^2$. Since $p < 1/2$, there is a drift to the left in the Markov chain corresponding to this random walk. Thus, there is a positive probability that S_n gets negative values and wanders off to $-\infty$ without ever reaching the value of 1. Hence the probability of $\{\bigcup_{n\geq 1}\{S_n \geq 1\}\}$ should be less than 1 and it is $p/(1-p)$.

Part d) For $p = 1/2$, show that the quadratic equation in part (c) has a double root at 1, and thus $\Pr\left\{\bigcup_{n\geq 1}\{S_n \geq 1\}\right\} = 1$. Note: this is the very peculiar case explained in the section on Wald's equality.

Solution: When $p = 1/2$, $p/(1-p) = 1$ and so both the roots of the quadratic function are 1. Although $\Pr\left\{\bigcup_{n\geq 1}\{S_n \geq 1\}\right\} = 1$ in this case, the expected time to reach the value of 1 for this Random walk is infinite, the corresponding birth-death Markov chain is null-recurrent and that is the reason for the peculiar behavior of that.

Part e) For $p < 1/2$, show that $p/(1-p) = \exp(-r*)$ where $r*$ is the unique positive root of $g_X(r) = 1$ where $g(r) = \mathbb{E}[e^{rx}]$.

Solution: The moment generating function is $g_X(r) = pe^r + (1-p)e^{-r}$. The solution to the equation $g_X(r) = 1$ satisfies $p+(1-p)e^{-2r*} = e^{-r*}$. Setting $z = \exp(-r*)$, we have $p+(1-p)z^2 = z$.This equation has two solutions $z = 1$ or $z = p/(1-p)$. $z = 1$ corresponds to $r* = 0$ and $z = p/(1-p)$ corresponds to the unique positive solution of the equation where $\exp(-r*) = p/(1-p)$. (Since $p < 1/2$, $p/(1-p) < 1$ and $r* > 0$).

<u>Exercise 7.3:</u>
A G/G/1 queue has a deterministic service time of 2 and interarrival times that are 3 with probability p and 1 with probability $1-p$.

Part a) Find the distribution of W_1, the wait in queue of the first arrival after the beginning of a busy period.

Solution: We know that $W_1 = \max(Y_0 - X_1, 0)$. $Y_0 = 2$ with probability 1, $X_1 = 3$ with probability p and $X_1 = 1$ with probability $1-p$. So $Y_0 - X_1 = -1$ with probability p and is equal to 1 with probability $1-p$. Thus, $W_1 = 0$ with probability p and is equal to 1 with probability $1-p$.

Part b) Find the distribution of W_∞, the steady state wait in queue.

Solution: We define $U_n = Y_{n-1} - X_n$ and similarly to part (a), it can be proved that U_n's are IID random variable that are equal to 1 with probability $1-p$ and are equal to -1 with probability p. Similar to the analysis done in section 7.2 of the notes, we know that:

$$W_n = \max(0, Z_1^n, Z_2^n, \cdots, Z_n^n)$$

Where $Z_i^n = U_n + U_{n-1} + \cdots + U_{n-i+1}$ for $1 \leq i \leq n$. Thus, Z_i^n is a random walk of $U_n, U_{n-1}, \cdots, U_{n-i+1}$ which are IID random variables. Hence,

$$\begin{aligned}
\Pr\{W_\infty \geq k\} &= \lim_{n\to\infty} \Pr\{\max(0, Z_1^n, Z_2^n, \cdots, Z_n^n) \geq k\} \\
&= \Pr\left\{\bigcup_{n\geq 1} Z^n \geq k\right\} \\
&= \Pr\left\{\bigcup_{n\geq 1} Z^n \geq 1\right\}^k
\end{aligned}$$

Thus, $\Pr\{W_\infty \geq k\} = \left(\frac{1-p}{p}\right)^k$ if $p > 1/2$ and $\Pr\{W_\infty \geq k\} = 1$ if $p < 1/2$.

Exercise 7.4:

A sales executive hears that one of his sales people is routing half of his incoming sales to a competitor. In particular, arriving sales are known to be Poisson at rate one per hour. According to the report (which we view as hypothesis 1), each second arrival is routed to the competition; thus under hypothesis 1 the interarrival density for successful sales is $f(y|H_1) = ye^{-y}$; $y \geq 0$. The alternative hypothesis (H_0) is the rumor is false and the interarrival density for successful sales is $f(y|H_0) = e^{-y}$; $y \geq 0$. Assume that, a priori, the hypotheses are equally likely. The executive, a recent student of stochastic processes, explores various alternatives for choosing between the hypotheses; he can only observe the times of successful sales however.

Part a) Starting with a successful sale at time 0, let S_i be the arrival time of the i-th subsequent successful sale. The executive observes $S_1, S_2, \cdots, S_n$ ($n \geq 1$) and chooses the maximum aposteriori probability hypothesis given this data. Find the joint density

$f(S_1, S_2, \cdots, S_n|H_1)$ and $f(S_1, S_2, \cdots, S_n|H_0)$ and give the decision rule.

Solution: Let's define Y_i to be the interarrival time of the ith successful sale (i.e., $Y_i = S_i - S_{i-1}$) and $Y_1 = S_1$. We know that Y_i's are IID and their distribution conditional on each hypothesis is given.

Thus,

$$
\begin{aligned}
f(Y_1, Y_2, \cdots, Y_n|H_1) &= f(Y_1|H_1)f(Y_2|H_1)\cdots f(Y_n|H_1) \\
&= \prod_{i=1}^{n} f(Y_i|H_1) \\
&= \prod_{i=1}^{n} Y_i e^{-Y_i} \\
&= e^{-(Y_1+\cdots+Y_n)} \prod_{i=1}^{n} Y_i \\
&= e^{-S_n} \prod_{i=1}^{n} Y_i
\end{aligned}
$$

Based on the definition of Y_i, $f(S_1, S_2, \cdots, S_n|H_1) = e^{-S_n}\prod_{i=1}^{n}(S_i - S_{i-1})$ where S_0 is defined to be 0.

Similarly,

$$f(Y_1, Y_2, \cdots, Y_n|H_0) = e^{-S_n}$$

Thus, $f(S_1, S_2, \cdots, S_n|H_0) = e^{-S_n}$. The optimal decision making rule, assuming the equiprobable priors for the hypotheses is a thresholding on likelihood ratio function. If $\Lambda(S_1, \cdots, S_n) > 1$ then the hypothesis H_1 is chosen, otherwise hypothesis H_0 is chosen.

$$
\begin{aligned}
\Lambda(S_1, \cdots, S_n) &= \frac{f(S_1, S_2, \cdots, S_n|H_1)}{f(S_1, S_2, \cdots, S_n|H_0)} \\
&= \frac{e^{-S_n}\prod_{i=1}^{n}(S_i - S_{i-1})}{e^{-S_n}} \\
&= \prod_{i=1}^{n}(S_i - S_{i-1})
\end{aligned}
$$

So if $\prod_{i=1}^{n}(S_i - S_{i-1}) \geq 1$, hypothesis H_1 is chosen and if $\prod_{i=1}^{n}(S_i - S_{i-1}) < 1$, hypothesis H_0 is chosen.

Part b) This is the same as part (a) except that the system is in steady state at time 0 (rather than starting with a successful sale). Find the density of S_1 (the time of the first arrival after time 0) conditional on H_0 and H_1. What is the decision rule now after observing $S_1, S_2, \cdots, S_n$.

Solution: The only difference with part (a) is the distribution of the first arrival. The consecutive arrivals are going to have the same distribution as they had in part (a).

Under hypothesis H_0, due to the memoryless property of the exponential distribution, the remaining time to the next arrival is also going to be exponentially distributed. So $f(Y_1|H_0) = e^{-Y_1}$; $Y_1 \geq 0$, and $f(S_1, S_2, \cdots, S_n|H_0) = e^{-S_n}$.

Under the hypothesis H_1, since the system is in steady state, two cases are possible at time 0. Either the first arrival will be routed to the competition or it will be a successful sale.These two events are equiprobable. If the first sale is supposed to be successful sale, its arrival time should be distributed exponentially (as under hypothesis H_1). If it is going to be routed, the arrival time of the first successful job is the sum of two arrival times, each distributed exponentially (the first one will be routed and the second one will be successful). This will be the same distribution that the later arrival times have under hypothesis H_1. Thus, $f(Y_1 = y|H_1) = 1/2e^{-y} + 1/2ye^{-y}$.

Based on similar analysis to part (a), $f(S_1, S_2, \cdots, S_n|H_1) = e^{-S_n} \prod_{i=1}^{n} (S_i - S_{i-1})$ and

$$
\begin{aligned}
f(S_1, S_2, \cdots, S_n|H_1) &= f(Y_1 = S_1|H_1) f(S_2, \cdots, S_n|H_1, S_1) \\
&= f(Y_1 = S_1|H_1) f(Y_2 = S_2 - S_1, \cdots, Y_n = S_n - S_{n-1}|H_1) \\
&= \left[1/2e^{-S_1} + 1/2S_1e^{-S_1}\right] \prod_{i=2}^{n} (S_i - S_{i-1}) e^{-(S_i - S_{i-1})} \\
&= \frac{1}{2} e^{-S_n} [1 + S_1] \prod_{i=2}^{n} (S_i - S_{i-1})
\end{aligned}
$$

So the likelihood ration function will be:

$$
\begin{aligned}
\Lambda(S_1, \cdots, S_n) &= \frac{f(S_1, S_2, \cdots, S_n|H_1)}{f(S_1, S_2, \cdots, S_n|H_0)} \\
&= \frac{1/2e^{-S_n}(1 + S_1) \prod_{i=2}^{n} (S_i - S_{i-1})}{e^{-S_n}} \\
&= 1/2(1 + S_1) \prod_{i=2}^{n} (S_i - S_{i-1})
\end{aligned}
$$

Again, if $\Lambda(S_1, \cdots, S_n) \geq 1$, hypothesis H_1 is chosen, and if $\Lambda(S_1, \cdots, S_n) < 1$, hypothesis H_0 is chosen.

Part c)

This is the same as part (b) except rather than observing n successful sales, the successful sales up to some given time t are observed. Find the probability, under each hypothesis, that the first successful sale occurs in $(s_1, s_1 + \Delta]$, the second in $(s_2, s_2 + \Delta]$, $\cdots$, and the last in $(s_{N(t)}, s_{N(t)} + \Delta]$ (assume Δ very small). What is the decision rule now?

Solution:
Suppose the observed data is $s_1, s_2, \cdots, s_n$ where n is the observed value of $N(t)$. Under H_0, the probability of arrivals in $(s_1, s_1+\Delta), \cdots, (s_n, s_n+\Delta)$ (Δ very small) is the probability that the first n arrivals are in these intervals times the probability of no arrival from $s_{n+\Delta}$ to t. This is $\Delta \exp(-s_1)\Delta \exp(-s_2+s_1)\cdots\Delta\exp(-s_n+s_{n-1})\exp(-t+s_n) = \Delta^n \exp(-t)$. Similarly, for H_1, the probability of an arrival in $(s_1, s_1+\Delta)$ is $\Delta(1+s_1)\exp(-s_1)/2$. The probability of each subsequent arrival i is $\Delta(s_i - s_{i-1})\exp(-s_i + s_{i-1})$. Finally, the probability of no arrival in (s_n, t) is $\int_{x>(t-s_n)} x\exp(-x)\,dx = (t - s_n + 1)\exp(-t+s_n)$. Thus the probability of arrivals in $(s_1, s_1+\Delta), \cdots, (s_n, s_n+\Delta)$ is $\Delta^n(1+s_1)/2(s_2-s_1)\cdots(s_n - s_{n-1})(t - s_n + 1)\exp(-t)$. Taking the ratio of these probabilities, we choose H_1 if

$$\frac{s_1+1}{2}\left[\prod_{i=1}^{n}(s_i - s_{i-1})\right](t - s_n + 1) > 1$$

.

We choose H_0 if this is strictly less than 1, and we don't care if it is equal to 1.

Exercise 7.5:

For the hypothesis testing problem of Section 7.3., assume that there is a cost C_0 of choosing H_1 when H_0 is correct, and a cost C_1 of choosing H_0 when H_1 is correct. Show that a threshold test minimizes the expected cost using the threshold $\eta = (C_1 p_1)/(C_0 p_0)$.

Solution: For a given sequence of observations $\mathbf{y}$, if we select $\hat{h} = 0$, an error is made if $h = 1$. The probability of this even is $\Pr\{H = 1|\mathbf{y}\}$ and the expected cost will be $c_1 \Pr\{H = 1|\mathbf{y}\}$.

If we select $\hat{h} = 1$, an error is made if $h = 0$. The probability of this even is $\Pr\{H = 0|\mathbf{y}\}$ and the expected cost will be $c_0 \Pr\{H = 0|\mathbf{y}\}$.

In order to minimize the expected loss, the decision rule should choose $\hat{h} = 1$ if $c_0 \Pr\{H = 0|\mathbf{y}\} < c_1 \Pr\{H = 1|\mathbf{y}\}$. Thus, $\hat{h} = 1$ if:

$$\begin{aligned}
c_0 \Pr\{H = 0|\mathbf{y}\} &< c_1 \Pr\{H = 1|\mathbf{y}\} \\
\frac{\Pr\{H = 0|\mathbf{y}\}}{\Pr\{H = 1|\mathbf{y}\}} &< \frac{c_1}{c_0} \\
\frac{p_0 \Pr\{\mathbf{y}|H = 0\}}{p_1 \Pr\{\mathbf{y}|H = 1\}} &< \frac{c_1}{c_0} \\
\frac{\Pr\{\mathbf{y}|H = 0\}}{\Pr\{\mathbf{y}|H = 1\}} &< \frac{p_1 c_1}{p_0 c_0}
\end{aligned}$$

<u>Exercise 7.10:</u> Consider a random walk with thresholds $\alpha > 0$, $\beta < 0$. We wish to find $\Pr\{S_J \geq \alpha\}$ in the absence of a lower threshold. Use the upper bound in (7.42) for the probability that the random walk crosses α before β.

Part a) Given that the random walk crosses β first, find an upper bound to the probability that α is now crossed before a yet lower threshold at 2β is crossed.

Solution: Let J_1 be the stopping trial at which the walk first crosses either α or β. Let J_2 be the stopping trial at which the random walk first crosses either α or 2β (assuming the random walk continues forever rather than actually stopping at any stopping trial. Note that if $S_{J_1} \geq \alpha$, then $S_{J_2} = S_{J_1}$, but if $S_{J_1} \leq \beta$, then it is still possible to have $S_{J_2} \geq \alpha$. In order for this to happen, a random walk starting at trial J_1 must reach a threshold of $\alpha - S_{J_1}$ before reaching $2\beta - S_{J_1}$. Putting this into equations,

$$\Pr\{S_{J_2} \geq \alpha\} = \Pr\{S_{J_1} \geq \alpha\} + \Pr\{S_{J_2} \geq \alpha \mid S_{J_1} \leq \beta\} \Pr\{S_{J_1} \leq \beta\}$$
$$\Pr\{S_{J_2} \geq \alpha \mid S_{J_1} \leq \beta\} \leq \exp[r^*(\alpha - \beta)],$$

where the latter equation upper bounds the probability that the RW starting at trial J_1 reaches $\alpha - S_{J_1}$ before $2\beta - S_{J_1}$, given that $S_{J_1} \leq \beta$.

Part b) Given that 2β is crossed before α, upperbound the probability that α is crossed before a threshold at 3β. Extending this argument to successively lower thresholds, find an upper bound to each successive term, and find an upper bound on the overall probability that α is crossed. By observing that β is arbitrary, show that (7.42) is valid with no lower threshold.

Solution: Let J_k for each $k \geq 1$ be the stopping trial for crossing α before $k\beta$. By the same argument as above,

$$\begin{aligned} \Pr\left\{S_{J_{k+1}} \geq \alpha\right\} &= \Pr\{S_{J_k} \geq \alpha\} + \Pr\left\{S_{J_{k+1}} \geq \alpha \mid S_{J_k} \leq k\beta\right\} \Pr\{S_{J_k} \leq k\beta\} \\ &\leq \Pr\{S_{J_k} \geq \alpha\} + \exp[r^*(\alpha - k\beta)], \end{aligned}$$

Finally, let J_∞ be the defective stopping time at which α is first crossed. We see from above that the event $S_{J_\infty} > \alpha$ is the union of the the events $S_{J_k} \geq \alpha$ over all $k \geq 1$. We can upperbound this by

$$\begin{aligned} \Pr\{S_{J_\infty} \geq \alpha\} &\leq \Pr\{S_{J_1} \geq \alpha\} + \sum_{k=1}^{\infty} \Pr\left\{S_{J_{k+1}} \geq \alpha \mid S_{J_k} \leq k\beta\right\} \\ &\leq \exp[r^*\alpha] \frac{1}{1 - \exp[r^*\beta]} \end{aligned}$$

Since this is true for all $\beta < 0$, it is valid in the limit $\beta \to -\infty$, yielding $e^{-r^*\alpha}$.

The reason why we did not simply take the limit $\beta \to -\infty$ in the first place is that such a limit would not define a defective stopping rule as any specific type of limit. The approach here was to define it as a union of non-defective stopping rules.