

**MASSACHUSETTS INSTITUTE OF TECHNOLOGY**  
Department of Electrical Engineering and Computer Science

6.262 Discrete Stochastic Processes  
Midterm Quiz  
April 6, 2010

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There are 5 questions, each with several parts. If any part of any question is unclear to you, ***please ask***.

The blue books are for scratch paper only. Don't hand them in. Put your final answers in the white booklets and briefly explain your reasoning for every question. **Please put your name on each white booklet you turn.**

Few questions require extensive calculations and most require very little, provided you pick the right tool or model in the beginning. The best approach to each problem is to first think carefully over what you've learned and decide precisely what tool fits best - before putting pencil to paper.

**Partial Credit**

We will give partial credit if you present your thinking in a clear way we can understand (and your thinking is at least partially correct), but otherwise not. If you model a problem using a tool that requires significant computation, it is best to first give the model explicitly and indicate how you will use the results of the computation to determine the final answer. This approach will help you receive fair credit if your computations aren't perfect.

**Time**

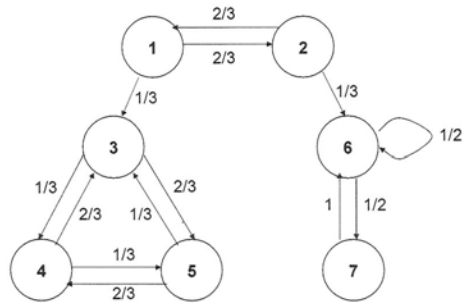
You will have at least 4 hours to finish the exam, and 30 – 60 minutes more if many of you request a bit more time.

**Useful Tables and Formulas**

These are given on the last pages of the quiz.

**Problem 1** (19 pts.)

Consider the following finite-state Markov Chain.



(5) a) Identify all the classes present in the chain and the states belonging to each class. Find the period of each class and determine whether the class is transient or recurrent.

b) Let  $p_{i,j}(n)$  denote the probability of the process ending up in state  $j$  in  $n$  transitions, conditioned on the fact that the process started in state  $i$ . In other words,  $p_{i,j}(n) = P(X_n = j | X_0 = i)$ . Compute the value of each of the limits below, or else explain briefly why it does not exist.

(2) i)  $\lim_{n \rightarrow \infty} p_{1,5}(n)$ .

(2) ii)  $\lim_{n \rightarrow \infty} p_{1,7}(n)$ .

(2) iii)  $\lim_{n \rightarrow \infty} p_{1,2}(n)$ .

(2) iv)  $\lim_{n \rightarrow \infty} p_{4,5}(n)$ .

(6) c) Let  $P = [p_{i,j}]$  be the transition matrix for this chain. Find **all** the possible steady state vectors for this chain, i.e., find **all** vectors  $\pi = [\pi_1, \pi_2, \dots, \pi_7]$  with the properties that  $\pi_1 + \pi_2 + \dots + \pi_7 = 1$ ,  $0 \leq \pi_1, \dots, \pi_7 \leq 1$  and  $\pi P = \pi$ .

## Solution

- a) There are three classes present:  $\{1, 2\}$ ,  $\{3, 4, 5\}$ , and  $\{6, 7\}$ . Class  $\{1, 2\}$  is transient and periodic with period two, while  $\{3, 4, 5\}$  and  $\{6, 7\}$  are both aperiodic and recurrent.
- b) Since the class  $\{1, 2\}$  is transient, we have that  $\lim_{n \rightarrow \infty} p_{1,2}(n) = 0$ , as the process eventually gets trapped in one of the two recurrent classes.

Since both recurrent classes are aperiodic, once the process enters a recurrent class containing state  $i$ , the long-term probability of being in state  $i$  equals the corresponding steady-state probability  $\pi_i$ . Considering first the class  $\{3, 4, 5\}$ , by symmetry in the transition probabilities, the long-term probability of the process being in a particular state is the same for every state in the class. Thus,  $\lim_{n \rightarrow \infty} p_{4,5}(n) = 1/3$ . Similarly, looking at  $\{6, 7\}$ , the simple form of the chain allows us to conclude that  $\lim_{n \rightarrow \infty} p_{7,7}(n) = 1/3$ . (If you don't see it immediately, then let  $\pi_6$  be the steady state probability of being in state 6 and  $\pi_7$  be that corresponding to state 7, where both are conditioned on the process starting out in the recurrent class  $\{6, 7\}$ . Conditioning on the previous transition:  $\pi_6 = (1/2)\pi_6 + \pi_7$ . But also  $\pi_6 + \pi_7 = 1$ , and the result follows solving the two equations.)

Finally, since states 1 and 2 are transient, let  $q_{1,\{3,4,5\}}$  and  $q_{2,\{3,4,5\}}$  denote the probabilities of eventually being trapped in  $\{3, 4, 5\}$  starting from states 1 and 2 respectively.

$$\begin{aligned} q_{1,\{3,4,5\}} &= \frac{1}{3}(1) + \frac{2}{3}q_{2,\{3,4,5\}} \\ q_{2,\{3,4,5\}} &= \frac{1}{3}(0) + \frac{2}{3}q_{1,\{3,4,5\}} \end{aligned}$$

It follows that

$$q_{1,\{3,4,5\}} = \frac{3}{5} \quad q_{2,\{3,4,5\}} = \frac{2}{5}.$$

Write

$$p_{1,5}(n) = p_{1,5|A_{1,n}}(n)\mathbf{P}(A_{1,n}) + p_{1,5|B_{1,n}}(n)\mathbf{P}(B_{1,n}) + p_{1,5|C_{1,n}}(n)\mathbf{P}(C_{1,n}),$$

where the events  $A_{1,n}$ ,  $B_{1,n}$ ,  $C_{1,n}$  are as follows:

$$\begin{aligned} A_{n,1} &= \{X_n \in \{3, 4, 5\} \mid X_0 = 1\} \\ B_{n,1} &= \{X_n \in \{1, 2\} \mid X_0 = 1\} \\ C_{n,1} &= \{X_n \in \{6, 7\} \mid X_0 = 1\} \end{aligned}$$

By our previous calculations,  $\lim_{n \rightarrow \infty} \mathbf{P}(A_{1,n}) = 3/5$ ,  $\lim_{n \rightarrow \infty} \mathbf{P}(B_{1,n}) = 0$ ,  $\lim_{n \rightarrow \infty} \mathbf{P}(C_{1,n}) = 1 - 3/5 = 2/5$ . Moreover,  $\lim_{n \rightarrow \infty} p_{1,5|A_{1,n}}(n) = 1/3$  and  $p_{1,5|C_{1,n}}(n) = 0$  for all  $n$ . Thus,  $\lim_{n \rightarrow \infty} p_{1,5}(n) = (1/3)(3/5) = 1/5$ . Similarly,  $\lim_{n \rightarrow \infty} p_{1,7}(n) = (1/3)(2/5) = 2/15$ .

To recap:

- i)  $\lim_{n \rightarrow \infty} p_{1,5}(n) = 1/5$ .
- ii)  $\lim_{n \rightarrow \infty} p_{1,7}(n) = 2/15$ .
- iii)  $\lim_{n \rightarrow \infty} p_{1,2}(n) = 0$ .
- iv)  $\lim_{n \rightarrow \infty} p_{4,5}(n) = 1/3$ .

c) Let  $\pi = [\pi_1, \pi_2, \dots, \pi_7]$  be such that  $\pi P = \pi$ . Writing out the corresponding equations, observe that

$$\pi P = \pi \quad \implies \quad \pi_1 = \pi_2 \frac{2}{3}, \quad \pi_2 = \pi_1 \frac{2}{3}.$$

Thus,  $\pi_1 = \pi_2 = 0$ .

Next, that  $\pi P = \pi$  also implies that

$$\begin{aligned} \pi_3 &= \pi_4 \frac{2}{3} + \pi_5 \frac{1}{3} \\ \pi_4 &= \pi_3 \frac{1}{3} + \pi_5 \frac{2}{3} \\ \pi_3 &= \pi_4 \frac{1}{3} + \pi_5 \frac{2}{3} \end{aligned}$$

Let  $\alpha \in [0, 1]$  be given by  $\alpha = \pi_3 + \pi_4 + \pi_5$ . It follows that  $\pi_3 = \pi_4 = \pi_5 = \alpha/3$ . (Equivalently, here we could have also noted that the system is under-determined, with one degree of freedom. Letting say  $x_1 = \tilde{\alpha}$  for some  $\tilde{\alpha} \in [0, 1]$  would yield the same conclusion in the end.)

Finally, that  $\pi P = \pi$  also implies that

$$\begin{aligned} \pi_6 &= \pi_6 \frac{1}{2} + \pi_7 \\ \pi_7 &= \pi_6 \frac{1}{2} \end{aligned}$$

Let  $\beta \in [0, 1]$  be given by  $\beta = \pi_6 + \pi_7$ . It follows that  $\pi_6 = 2\beta/3$ ,  $\pi_7 = \beta/3$ .

Since  $\pi_1 = \pi_2 = 0$ ,  $\alpha + \beta = 1$ . Therefore,

$$\pi = \alpha[0, 0, 1/3, 1/3, 1/3, 0, 0] + (1 - \alpha)[0, 0, 0, 0, 0, 2/3, 1/3] \quad \text{for } \alpha \in [0, 1].$$

Let  $\pi' = [0, 0, 1/3, 1/3, 1/3, 0, 0]$  and  $\pi'' = [0, 0, 1/3, 1/3, 1/3, 0, 0]$ . Since  $\pi' P = \pi'$  and  $\pi'' P = \pi''$ , for any choice of  $\alpha$  we have

$$(\alpha\pi' + (1 - \alpha)\pi'')P = \alpha\pi'P + (1 - \alpha)\pi''P = \alpha P + (1 - \alpha)P = P.$$

Moreover, any choice of  $\alpha \in [0, 1]$  yields a vector  $\pi$  such that  $0 \leq \pi_i \leq 1$  for  $i = 1, \dots, 6$  and  $\sum_i \pi_i = 1$ . Therefore, the set of all probability vectors  $\pi$  such that  $\pi P = P$  is given by:

$$\pi = \alpha[0, 0, 1/3, 1/3, 1/3, 0, 0] + (1 - \alpha)[0, 0, 0, 0, 0, 2/3, 1/3] \quad \text{for } \alpha \in [0, 1]$$

## Problem 2

Consider a car ferry that holds some integer number  $k$  of cars and carries them across a river. The ferry business has been good, but customers complain about the long wait for the ferry to fill up. The cars arrive according to a renewal process. The ferry departs immediately upon the arrival of the  $k$ -th customer and subsequent ferries leave immediately upon the arrival of the  $2k$ -th customer, the  $3k$ -th customer, etc.

a) The IID inter-arrival times of the cars have mean  $\bar{X}$ , variance  $\sigma^2$  and moment generating function  $g_X(r)$ . Does the sequence of departure times of the ferries form a renewal process? Explain carefully.

**Solution:** Yes, the ferry departure times form a renewal process. The reason is that the  $\ell$ -th ferry departure is immediately after the  $(k\ell)$ -th customer arrival. The time from the  $\ell$ -th to  $\ell+1$  ferry departure is the time from the  $(k\ell+1)$ -st to  $((k+1)\ell)$ -th customer arrival, which is clearly independent of all previous customer arrival times and therefore of all previous ferry departure times.

b) Find the expected time that a randomly chosen customer waits from arriving at the ferry terminal until departure of its ferry. As part of your solution, please give a reasonable definition of the expected waiting time for a randomly chosen customer, and please first solve this problem explicitly for the cases  $k = 1$  and  $k = 2$ .

**Solution:** For  $k = 1$ , the ferry leaves immediately when a customer arrives, so the expected waiting time for each customer is 0. For  $k = 2$ , odd numbered customers wait for the following even numbered customer, and even number customers don't wait at all, so the average waiting time over customers (which we take as the only sensible definition of expected waiting time) is  $\bar{X}/2$ .

We next find the expected waiting time, averaged over customers, for the  $\ell$ th ferry. To simplify notation, we look at the first ferry. The average expected wait over the  $k$  customers is the sum of their expected waits divided by  $k$ . (Recall that this is true even if, as here, the waits of different customers are statistically dependent.) The expected wait of customer 1 is  $(k-1)\bar{X}$ , that of customer 2  $(k-2)\bar{X}$ , etc. Recall (or derive) that  $1+2+\dots+(k-1) = (k-1)k/2$ . Thus the expected wait per customer is  $(k-1)\bar{X}/2$ , which checks with the result for  $k = 1$  and  $k = 2$ .

c) Is there a 'slow truck' phenomenon here ? (This the phrase we used to describe the effect of large variance on the term  $E[X^2]/2E[X]$  in the steady state residual life or on the  $E[Z^2]$  term in the numerator of the Pollazcek-Khinchin formula.) Give a brief intuitive explanation.

**Solution:** Clearly, there is no 'slow truck' phenomenon for the ferry wait here since the answer depends only on  $k$  and  $\bar{X}$ . The reason is most evident for  $k = 1$ , where the wait is 0. The arrival of a car at the ferry terminal could have delayed by an arbitrary amount by a slow truck in *getting to* the ferry terminal, but is not delayed at all in *getting on* the ferry, since the slow truck took an earlier ferry. For larger  $k$ , a vehicle could be delayed by a later arriving truck, but at most  $k - 1$  vehicles could be delayed that way, while the  $E[X^2]$  effect arises from the potentially unbounded number of customers delayed by a slow truck.

d) In an effort to decrease waiting, the ferry managers institute a policy where the maximum interval between ferry departures is 1 hour. Thus a ferry leaves either when it is full or after one hour has elapsed, whichever comes first. Does the sequence of departure times of ferries that leave with a full load of cars constitute a renewal process? Explain carefully.

**Solution:** Yes. When a ferry departs with a full load of  $k$  cars, the auto arrival process restarts immediately with the wait for the first car for the next ferry. The subsequent sequence of departure times of all the partially full ferries up through the departure of the next full ferry is independent of the arrival times of cars to the previous full ferry and before. The times at which each successive ferry is entered by its first customer is also a renewal process since customer arrivals form a renewal process.

### **Problem 3**

The use of the various laws of large numbers with random variables that take huge values with tiny probabilities requires careful thought.

Except where exact answers are requested, your numerical answers need only be accurate to within  $\pm 1\%$ .)

Consider a discrete r.v.  $X$  with the PMF

$$p_X(-1) = (1 - 10^{-10})/2,$$

$$p_X(1) = (1 - 10^{-10})/2,$$

$$p_X(10^{12}) = 10^{-10}.$$

**a)** Find the mean and variance of  $X$ . Assuming that  $\{X_m; m \geq 1\}$  is an IID sequence with the distribution of  $X$  and that  $S_n = X_1 + \dots + X_n$  for each  $n$ , find the mean and variance of  $S_n$ .

**Solution:**  $\bar{X} = 100$  and  $\sigma_X^2 = (10^{12} - 100)^2 \times 10^{-10} + (100+1)^2 \times \frac{(1-10^{-10})}{2} + (100-1)^2 \times \frac{(1-10^{-10})}{2} \approx 10^{14} - 2 \times 10^4 + 10^4 \approx 10^{14}$ .

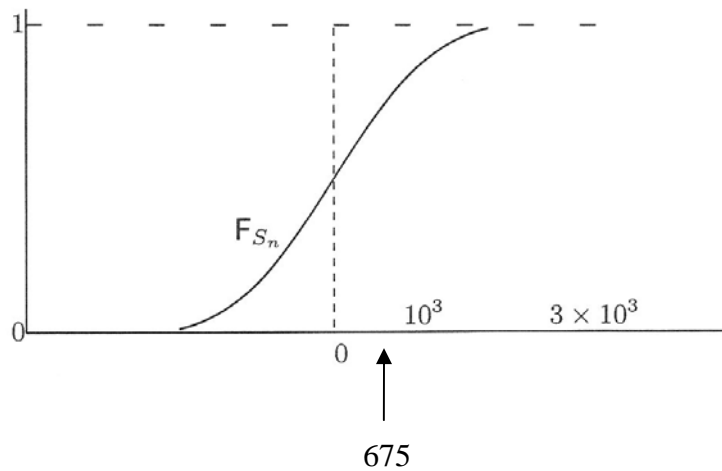
Thus  $\bar{S}_n = 100n$  and  $\sigma_{S_n}^2 \approx n \times 10^{14}$ .

**b)** Sketch the distribution function of  $S_n$  for  $n = 10^6$ . (You may plot it as if it were a continuous function, but use a linear scale for the x-axis.) Estimate the value of  $s$  to within  $\pm 1\%$  for which  $F_{S_n}(s) = 3/4$  and draw your sketch from  $-2s$  to  $2s$  on the horizontal axis.

**Solution:** We neglect the very small probability (about  $10^{-4}$ ) of an event with probability  $10^{-10}$  occurring one or more times in  $10^6$  tosses. (A more accurate derivation will be found in part c, below.) With this approximation,  $S_n$  simply has a binomial distribution, taking values  $+1$  and  $-1$ , each with probability  $\approx 1/2$ . We know from the central limit theorem for the binomial case that  $S_n$  will be approximately Gaussian with mean 0, variance  $10^6$  and standard deviation  $10^3$ .

Since one or more occurrences of  $10^{12}$  occur only with probability about  $10^{-4}$ , this possibility can be neglected, so the distribution function is approximately Gaussian with 3 sigma points at  $\pm 3 \times 10^3$ . From the table for the unit Gaussian  $\Phi$ ,  $F_\Phi(x) = 3/4$  when  $x = 0.675$ , so for  $n = 10^6$ ,  $F_{S_n}(s) = 3/4$  when  $s \approx 675$ .

To check for accuracy due to neglecting a probability of about  $10^{-4}$ , we notice that near  $x = 0.75$ ,  $\frac{d\Phi}{dx} \approx 0.31$ , so an error of around  $10^{-4}$  in probability would cause an error of around  $(1,000) \times (10^{-4}/.31) \approx 0.32$ , a negligible fraction of 675.



**c)** Again for  $n = 10^6$ , find an exact expression for  $1 - F_{S_n}(2 \times 10^6)$  when  $n = 10^6$  and give a simple numerical approximation of this value (a better approximation than 0).

**Solution:** For  $n = 10^6$ , the event  $\{S_n \leq 2 \times 10^6\}$  is the event {no outcomes of size  $10^{12}$  in  $10^6$  independent trials}, which has a probability:

$$F_{S_{10^6}}(2 \times 10^6) = (1 - 10^{-10})^{10^6},$$

which, from the approximate formulas at the end of the quiz, is approximately  $e^{-10^{-4}} \approx 1 - 10^{-4}$ . Therefore,

$$1 - F_{S_{10^6}}(2 \times 10^6) \approx 1 - (1 - 10^{-4}) = 10^{-4}.$$

**d)** Now let  $n = 10^{10}$ . Give an exact expression for  $P(S_n \leq 10^{10})$  for  $n = 10^{10}$ , and find an approximate numerical value. Sketch the distribution function of  $S_n$  for  $n = 10^{10}$ . Scale the horizontal axis to include the points 0 near its left end and  $2 \times 10^{12}$  near its right end.



**Solution:** The event  $\{S_n \leq 10^{10}\}$  for  $n = 10^{10}$  trials is the event {no outcomes of size  $10^{12}$  in  $10^{10}$  independent trials}, Consider the PMF  $p_B(j)$  of the number  $B = j$  of occurrences of the value  $10^{12}$ . We have

$$p_B(j) = \binom{10^{10}}{j} p^j (1-p)^{10^{10}-j} \text{ where } p = 10^{-10},$$

$$p_B(0) = (1-p)^{10^{10}} = \exp\{10^{10} \ln[1-p]\} \approx \exp(-10^{10} p) = e^{-1},$$

(Note that the approximation  $(1-\varepsilon)^N \approx 1 - N\varepsilon + \frac{N(N-1)}{2} \varepsilon^2$

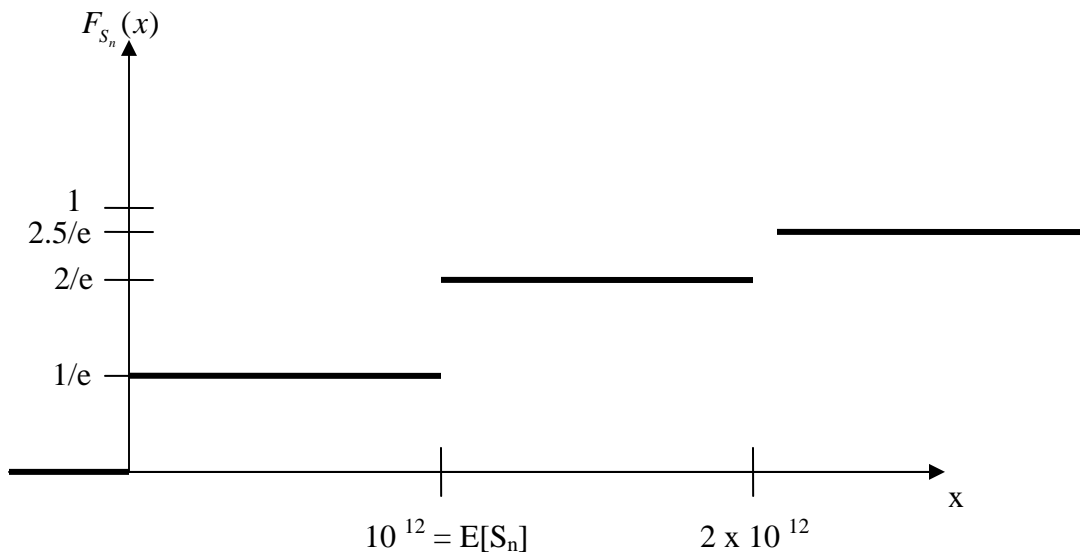
is not good unless  $N\varepsilon \ll 1$ ; it gives 1/2 in this case)

$$p_B(1) = 10^{10} p (1-p)^{10^{10}-1} = (1-p)^{10^{10}-1} \approx e^{-1},$$

$$p_B(2) = \binom{10^{10}}{2} p^2 (1-p)^{10^{10}-2} = \frac{(10^{10})(10^{10}-1)}{2} (10^{-10})^2 (1-10^{-10})^{10^{10}-2} \approx \frac{1}{2} e^{-1},$$

$$P(B \leq 2) \approx 2.5e^{-1}.$$

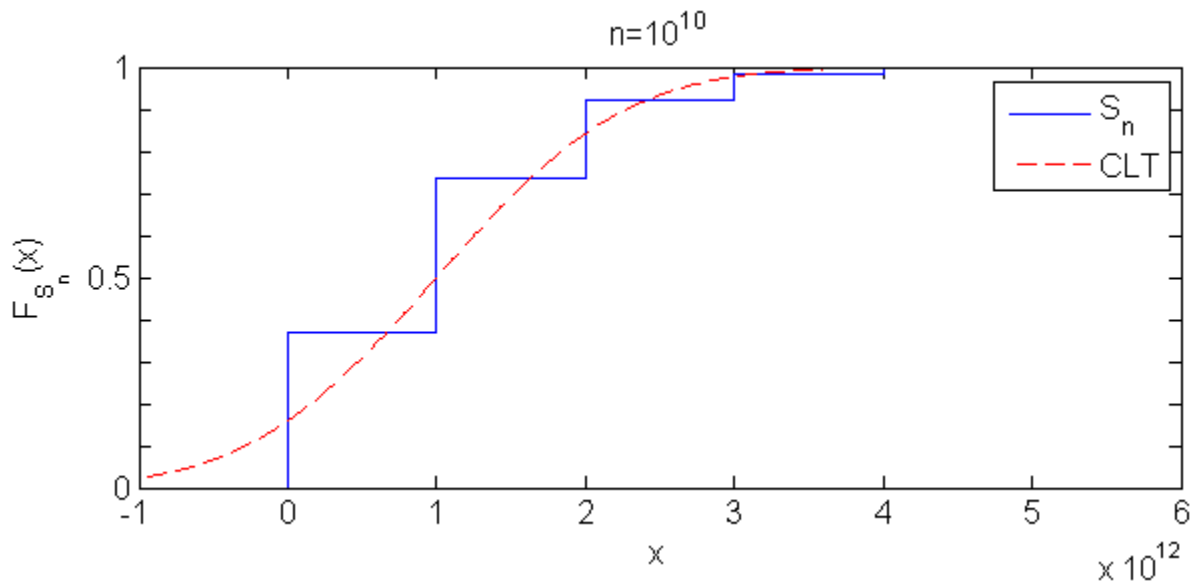
Conditional on  $B = j$ ,  $j = 0, 1$  or  $2$ ,  $S_n$  will be approximately Gaussian with mean  $10^{12} j$  and a relatively tiny standard deviation of  $10^5$ . Thus  $F_{S_n}(x)$  rises from 0 to  $e^{-1}$  over a range of  $x$  from about  $-3 \times 10^5$  to  $+3 \times 10^5$ . It then stays virtually constant up to about  $x = 10^{12} - 3 \times 10^5$ . It rises to  $2 \times e^{-1}$  by about  $x = 10^{12} + 3 \times 10^5$ . It stays virtually constant up to about  $x = 2 \times 10^{12} - 3 \times 10^5$  and rises to  $2.5e^{-1}$  by about  $x = 2 \times 10^{12} + 3 \times 10^5$ . When we sketch this, the rises in  $F_{S_{10^{10}}}(x)$  over a width of about  $6 \times 10^5$  look essentially vertical on a scale of  $2 \times 10^{12}$ , rising from 0 to  $1/e$  at 0, from  $1/e$  to  $2/e$  at  $10^{12}$  and from  $2/e$  to  $2.5/e$  at  $2 \times 10^{12}$ . There are smaller steps at larger values, but they would scarcely show up on this sketch.



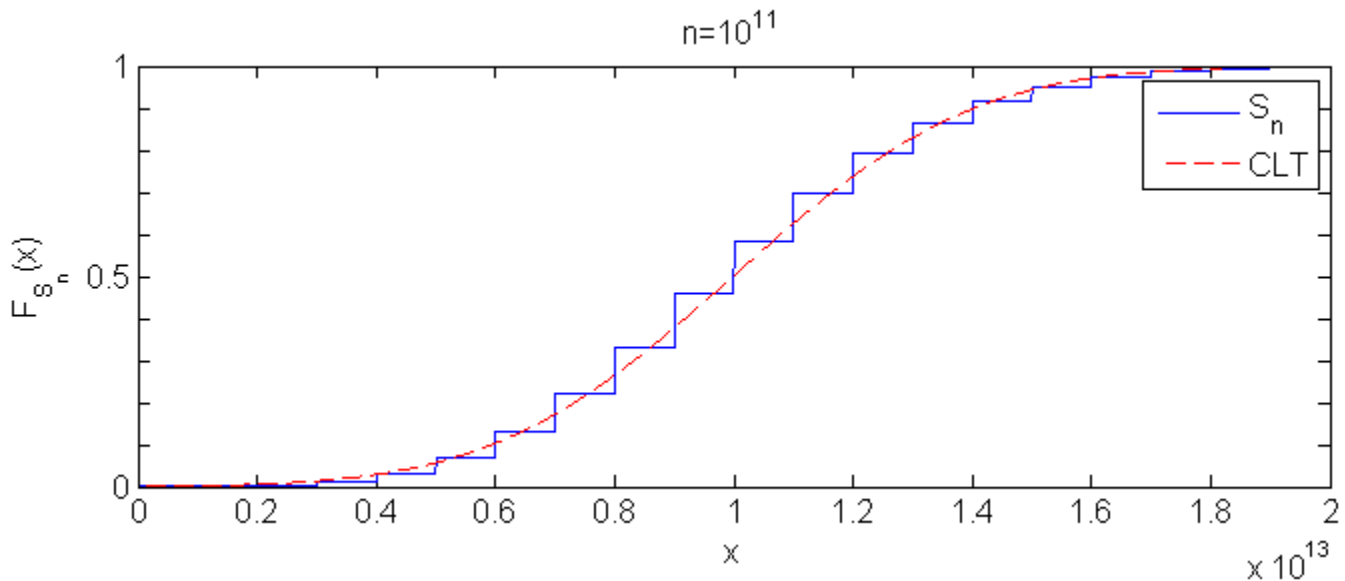
e) What is roughly (i.e., within an order of magnitude or so) the smallest value of  $n$  for which the central limit theorem would work well for this problem ?

**Solution:**

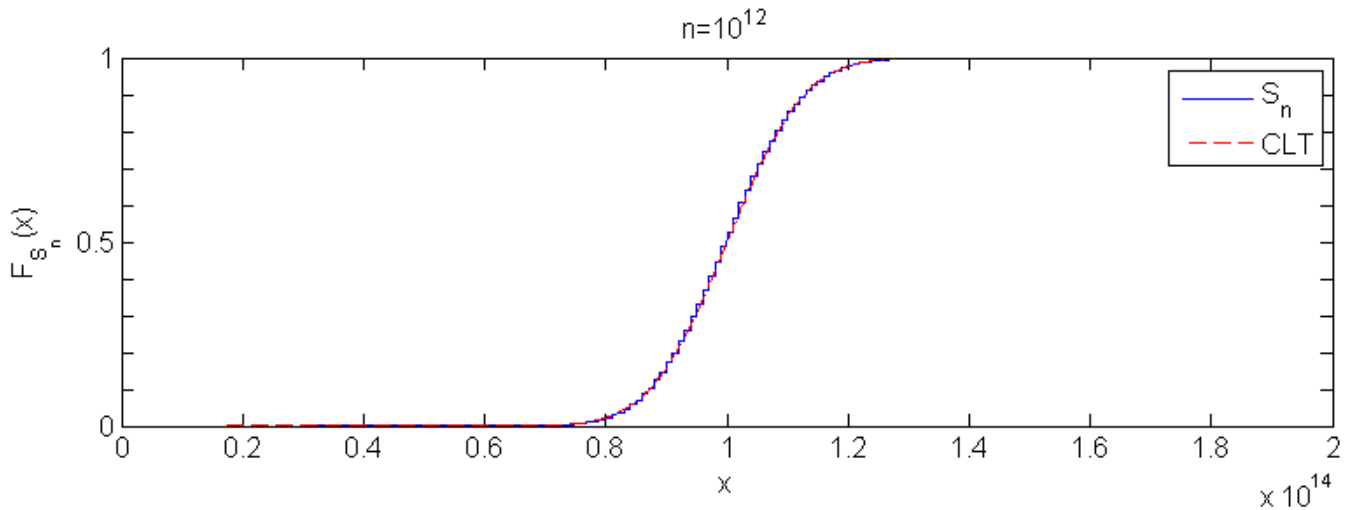
The plot below shows that, for this peculiar rv,  $S_n/n$  is not concentrated around its mean even for  $n=10^{10}$  and  $S_n/\sqrt{n}$  does not look Gaussian even for  $n=10^{10}$ . For this particular distribution,  $n$  has to be so large that  $B$ , the number of occurrences of  $10^{12}$  is larger, and this requires  $n$  to be significantly greater than  $10^{10}$ .



With  $n = 10^{11}$ ,  $S_n$  will have a mean of  $10^{13}$ , a variance of about  $10^{25}$ , and a standard deviation of about  $3.16 \times 10^{12}$ . This allows only 7 outcomes (i.e., 7 - 13 occurrences of  $10^{12}$ ) within 1 standard deviation of the mean, a very modest number to consider the Gaussian fit from the CLT to be reasonably accurate. See figure below.



With  $n = 10^{12}$ ,  $S_n$  will have a mean of  $10^{14}$ , a variance of about  $10^{26}$ , and a standard deviation of about  $10^{13}$ . This allows for 21 outcomes (i.e., 90 - 110 occurrences of  $10^{12}$ ) within 1 standard deviation of the mean, a better but still modest number to consider the Gaussian fit from the CLT to be reasonably accurate. See figure below.



With  $n = 10^{13}$ ,  $S_n$  would have a mean of  $10^{15}$ , a variance of about  $10^{27}$ , and a standard deviation of about  $3.16 \times 10^{13}$ . This allows for 63 outcomes (i.e., 969 - 1031 occurrences of  $10^{12}$ ) within 1 standard deviation of the mean, a better number to consider the Gaussian fit from the CLT to be reasonably accurate.

Thus some number in the range of  $n = 10^{11}$  to  $n = 10^{13}$  should be the smallest  $n$  that is adequate to accurately use the CLT. Somewhat more generally, suppose

$$p_X(-1) = (1-t)/2,$$

$$p_X(1) = (1-t)/2, \quad \text{where } t \text{ is tiny } (\ll 1) \text{ and } h \text{ is huge } (\gg 1).$$

$$p_X(h) = t,$$

$$\text{Then } E[X] = ht \qquad E[S_n] = nht,$$

$$\sigma_X^2 \approx [h^2 - ht]t \approx h^2t, \qquad \sigma_{S_n}^2 \approx nh^2t,$$

$$\sigma_X \approx h\sqrt{t} \qquad \sigma_{S_n} \approx h\sqrt{nt},$$

For the CLT to work well, we need  $E[\# \text{ huge outcomes}] = nt \gg 1$  &

the total distinct number of huge outcomes within  $1 \sigma_{S_n}$  of  $E[S_n] = \frac{2\sigma_{S_n}}{h} \approx 2\sqrt{nt} \gg 1$ .

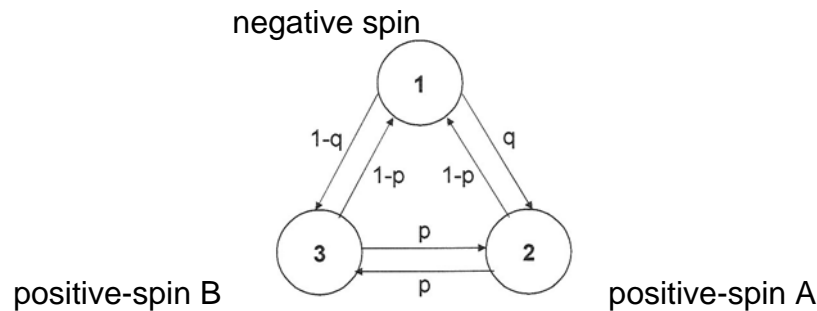
The latter condition determines the requirement on  $n$ .

This problem illustrates a common weakness of limit theorems. They say what happens as a parameter ( $n$  in this case) becomes sufficiently large, but it takes extra work to see what values of  $n$  satisfy that criterion.

**Problem 4** (20 pts)

A certain physical object emits three types of particles: negative-spin particles, positive-spin particles of type A and positive-spin particles of type B. A processing device records the times and the types of particle arrivals at the sensor.

Considering only the particle types (i.e. negative, positive A, positive B), the corresponding arrival process can be modeled as a finite-state Markov chain shown below, where State 1 is associated with a negative-spin particle, State 2 with a positive-spin particle of type A and State 3 with a positive-spin particle of type B. For example, after recording a negative-spin particle, the next arrival at the sensor will be a positive-spin type A arrival with probability  $q$  or a positive-spin type B arrival with probability  $1-q$ . Suppose that  $p \in (0, 1/2)$ ,  $q \in (0, 1/2)$ .



a) Let us consider several possible processes arising from the above setup.

(4) i) Do the arrivals in time of positive-spin particles of type A form a renewal process? A delayed renewal process?

(4) ii) Do the arrivals in time of positive-spin particles form a renewal process? A delayed renewal process?

(4) iii) Does the discrete-time process that records the spin orientations (i.e. positive vs. negative) of the incoming particles form a Bernoulli process?

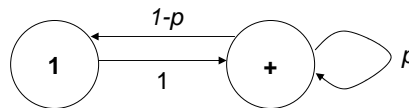
(4) iv) Does the discrete-time process that records the types (A vs. B) of incoming *positive-spin* particles form a Bernoulli process?

(4) b) Starting from State 1, let  $M_1$  denote the number of transitions until the chain returns to State 1. Find  $E(M_1)$  as well as the probability mass function of  $M_1$ .

## Solution

a) We are asked to consider several types of discrete stochastic processes. (That the processes are discrete was made additionally explicit during the exam.)

- i) The arrivals of type-A particles are given by arrivals to a recurrent state (state 2) of a finite-state Markov chain with a single recurrent class. Regardless of the starting state, the process will therefore eventually visit state 2 and return to it infinitely often with probability 1. Therefore, the first arrival epoch and the subsequent inter-arrival times (in terms of the numbers of transitions) of the type-A particles are finite with probability 1. Moreover, by the Markov property, the inter-arrival times form an IID sequence, with the possible exception of the first arrival which need not have the same distribution as the others. Therefore, conditioned on the chain starting in state 2, the process is a renewal process; otherwise, it is a delayed renewal process.
- ii) Here, we don't need to distinguish between states 2 and 3. Conveniently, the probability of accessing state 1 is the same from state 2 as from state 3. Therefore, as far as counting the number of particles between successive positive-spin particles is concerned, the process is probabilistically equivalent to the one illustrated below where states 2 and 3 have been merged into a single state and the hops between the two are modeled as a self-transition. (Note that symmetry was crucial for this equivalence.)



Since the "+" state is recurrent and belongs to a chain with a single recurrent class, arguments analogous to those of a) yield that the positive particles form a renewal process conditioned on the first particle being either positive A or positive B, or a delayed renewal process conditioned on the first particle being negative.

- iii) Given an arrival of a negative particle, the next particle is positive with probability 1. The process is therefore not Bernoulli (i.e. cannot be modeled as a sequence of coin tosses).
- iv) Given an arrival of a positive-spin particle of type A, the probability of the next positive particle being of type B is  $p + (1 - p)(1 - q)$ . On the other hand, conditioned on an arrival of a particle of type B, the probability that the next particle is of type B is  $(1 - p)(1 - q)$ . In a Bernoulli process, the probability that the next particle comes out B cannot depend on the previous particle. The two probabilities are equal only for  $p = 0$ , which falls outside of the range given. Therefore, the process is not Bernoulli. (We apologize for the hint, it should have asked you to consider the range of  $p$  and  $q$ .)
- b) Notice once again that due to symmetry, merging the positive-spin states as previously does not change the distribution of the number of transitions needed for the chain to return to state 1. The number of transitions from state "+" to 1 is given by a geometric random variable with mean  $1/(1 - p)$ . Thus,

$$P(M_1 = m) = (1 - p)p^{m-2} \quad \forall m = 2, 3, \dots \quad \text{and} \quad E(M_1) = \frac{1}{1 - p} + 1.$$

(Note that the added 1 includes the transition from state 1 to state "+".)

## Problem 5

Consider a first-come, first serve M/M/1 queue with a customer arrival rate  $\lambda$  and a service rate  $\mu$ , i.e.,

$$P(T_k > \tau) = e^{-\mu\tau}, \tau \geq 0,$$

where  $T_k$  is the service time for the k-th customer. Assume the queue is empty at  $t = 0$ , i.e., no customer is in the queue or in service at  $t = 0$ .

**a)** Find the expected total wait in queue plus service for the first customer. (No derivation or explanation required.)

**Solution:** Let  $W_n$  be the total wait in queue plus service of the nth customer. The first customer enters an empty system and goes immediately into service, with expected service time of

$$E[W_1] = 1/\mu.$$

**b)** Find the expected total wait in queue plus service for the second customer. (Please give complete calculation and briefly explain your reasoning.)

**Solution:** The second customer either encounters an empty system or else a system with one customer ahead of her in service. Therefore

$$\begin{aligned} E[W_2] &= E[W_2 \mid \text{system empty when cust. \#2 arrives}] P(\text{system empty when} \\ &\quad \text{cust. \#2 arrives}) \\ &+ \\ &\quad E[W_2 \mid \text{cust. \#1 in service when cust. \#2 arrives}] P(\text{cust. \#1 in} \\ &\quad \text{service when cust. \#2 arrives}) \end{aligned} =$$

(since exponential service time is memoryless)

$$\left(\frac{1}{\mu}\right) P(\text{system empty when cust. \#2 arrives}) + \left(\frac{2}{\mu}\right) P(\text{cust. \#1 in service when cust. \#2 arrives}).$$

To find the probabilities, begin at the arrival time of cust. #1 and split a Poisson process with rate  $(\lambda + \mu)$  into two processes, one with rate  $\lambda$  to model the arrival of cust. #2 and one with rate  $\mu$  to model the service completion time of cust. #1. The probability that cust. #2 arrives before cust. #1 has finished service is then  $(\lambda / (\lambda + \mu))$ , and therefore

$$E[W_2] = \left(\frac{1}{\mu}\right)(\mu / (\lambda + \mu)) + \left(\frac{2}{\mu}\right) (\lambda / (\lambda + \mu)) = \frac{2\lambda + \mu}{\mu(\lambda + \mu)}.$$

For the customer arrival process with rate  $\lambda$ , consider the age  $Z(t)$  of the interarrival interval at any time  $t \geq 0$ . (The questions below apply equally well to any Poisson process with rate  $\lambda$  starting at  $t = 0$ .) The first interarrival interval starts to  $t = 0$ .

**c)** Find the expected age  $E[Z(0)]$  and find (or recall)  $\lim_{t \rightarrow \infty} E[Z(t)]$ . (No explanation required.)

**Solution:**  $Z(0) = 0$  surely, so  $E[Z(0)] = 0$ .

$$\text{And } \lim_{t \rightarrow \infty} E[Z(t)] = \frac{\overline{X^2}}{2\overline{X}} = \frac{(\overline{X})^2 + \sigma_X^2}{2\overline{X}} = \frac{(1/\lambda)^2 + 1/\lambda^2}{2/\lambda} = \frac{1}{\lambda},$$

i.e., in steady state the Poisson process looks identical in forward or backward time.

**d)** Find the expected age  $E[Z(t)]$ ,  $\forall t \geq 0$ . (You can derive this from your answer to part e), or you can solve part d) separately. Please give a complete calculation and briefly explain your reasoning.)

**Solution:** Proceeding without first finding the distribution of  $Z(t)$ , we have

$$E[Z(t)] = \sum_{n=0}^{\infty} E[Z(t) | n \text{ arrivals happen in } [0,t)] \cdot P(n \text{ arrivals happen in } [0,t)) = \sum_{n=0}^{\infty} E[Z(t) | n \text{ arrivals happen in } [0,t)] \cdot e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

If  $n > 0$  arrivals occur in  $[0,t]$ , then for a Poisson process they are uniformly and independently distributed in  $[0,t]$ , so

$$E[Z(t) | n \text{ arrivals happen in } [0,t)] = E[S_1 | N(t) = n] = t/(n+1).$$

Therefore, since

$$E[Z(t) | 0 \text{ arrivals happen in } [0,t)] = t = t/(n+1),$$

$$E[Z(t)] = \sum_{n=0}^{\infty} E[Z(t) | n \text{ arrivals happen in } [0,t)] \cdot P(n \text{ arrivals happen in } [0,t)) = \sum_{n=0}^{\infty} \frac{t}{n+1} \cdot e^{-\lambda t} \frac{(\lambda t)^n}{n!} = te^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{(n+1)n!} = te^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{(n+1)!} = \frac{te^{-\lambda t}}{\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda t)^{n+1}}{(n+1)!} = \frac{te^{-\lambda t}}{\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{(n)!} = \frac{te^{-\lambda t}}{\lambda t} (e^{\lambda t} - 1) = \frac{1 - e^{-\lambda t}}{\lambda}.$$

**e)** Find  $F_{Z(t)}(z)$ ,  $\forall t \geq 0, \forall z \in [0,t]$ . (Please give a complete calculation and briefly explain your reasoning.)

**Solution:**  $P(Z(t) = t) = P(\text{no arrivals in } [0, t]) = e^{-\lambda t}$ .

**Approach #1 :** For  $z < t$ ,  $\{Z(t) > z\} \Leftrightarrow \{\text{no arrivals in } [t-z, t]\}$

$$P(\text{no arrivals in } [t-z, t]) = e^{-\lambda z}$$

$$\text{so } P(Z(t) \leq z) = \begin{cases} 1 - e^{-\lambda z}, & 0 \leq z < t \\ 1, & z \geq t \end{cases}$$

**Approach #2 :**

$P(0 \leq z < Z(t) < t) = P(\text{all } n > 0 \text{ arrivals in } [0, t] \text{ are earlier than } t-z) =$

$$\sum_{n=1}^{\infty} \left(\frac{t-z}{t}\right)^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} = e^{-\lambda t} \sum_{n=1}^{\infty} (t-z)^n \frac{(\lambda)^n}{n!} = e^{-\lambda t} (e^{\lambda(t-z)} - 1) = (e^{-\lambda z} - e^{-\lambda t}).$$

Therefore, for  $z < t$ ,

$$F_{Z(t)}(z) = P(Z(t) \leq z) = 1 - P(0 \leq z < Z(t) < t) - P(Z(t) = t) = 1 - (e^{-\lambda z} - e^{-\lambda t}) - e^{-\lambda t} = 1 - e^{-\lambda z},$$

and, for  $z = t$ ,

$$F_{Z(t)}(t) = P(Z(t) \leq t) = 1,$$

i.e.,

$$F_{Z(t)}(z) = \begin{cases} 1 - e^{-\lambda z}, & 0 \leq z < t \\ 1, & z \geq t. \end{cases}$$

Checking our answer to part d),

$$E[Z(t)] = \int_0^{\infty} (1 - F_{Z(t)}(z)) dz = \int_0^t (1 - F_{Z(t)}(z)) dz = \int_0^t e^{-\lambda z} dz = -\left(\frac{1}{\lambda}\right)e^{-\lambda z} \Big|_{z=0}^t = \frac{1 - e^{-\lambda t}}{\lambda} \quad (!!!).$$



### Possibly Useful Formula

For a large integer  $N$  and small  $\varepsilon$ , (i.e.,  $N \gg 1$  and  $0 < \varepsilon \ll 1$ ),

$$(1 - \varepsilon)^N \approx e^{-N\varepsilon}$$

which also approximately equals, if  $N\varepsilon \ll 1$  as well,

$$1 - N\varepsilon + \frac{N(N-1)}{2}\varepsilon^2 + \dots$$

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Spring 2011

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