6.262: Discrete Stochastic Processes 2/14/11

Lecture 4: Poisson (the perfect arrival process)

Outline:

- Review: Convergence and sequences of IID rv's.
- Arrival processes
- Poisson processes and exponential rv's.
- Stationary and independent increments
- The Erlang and Poisson distributions
- Alternate definitions of Poisson process
- Relation to Bernoulli process

Review: Convergence and sequences of IID rv's.

Def: A sequence $Z_1, Z_2, \dots,$ of random variables, converges in distribution to a random variable Z if $\lim_{n\to\infty} \mathsf{F}_{Z_n}(z) = \mathsf{F}_Z(z)$ at each z for which $\mathsf{F}_Z(z)$ is continuous.

Example: (CLT) If $X_1, X_2, \dots$ are IID with variance σ^2, $S_n = \sum_{i=1}^n X_i$, and $Z_n = (S_n - n\overline{X})/\sigma\sqrt{n}$, then $Z_1, Z_2, \dots$ converges in distribution to $\mathcal{N}(0,1)$.

Example: If $X_1, X_2, \dots,$ are IID with mean $\overline{X}$ and $S_n = \sum_{i=1}^n X_i$, then $\{S_n/n;\ n \geq 1\}$ converges in distribution to the deterministic rv $\overline{X}$.

Def: A sequence $Z_1, Z_2, \dots,$ of random variables converges in probability to a random variable Z <u>if</u> $\lim_{n\to\infty} \Pr\{|Z_n - Z| > \epsilon\} = 0$ for every $\epsilon > 0$ (alternatively, <u>if</u> for every $\epsilon > 0, \delta > 0$, $\Pr\{|Z_n - Z| > \epsilon\} \le \delta$ for all large enough n.)

Example: (WLLN) If $X_1, X_2, \dots,$ are IID with mean $\overline{X}$ and $S_n = \sum_{i=1}^{n} X_i$, then $\{S_n/n;\ n \ge 1\}$ converges in probability to the deterministic rv $\overline{X}$. (see Thms. 1.5.1 and 1.5.3 of text)

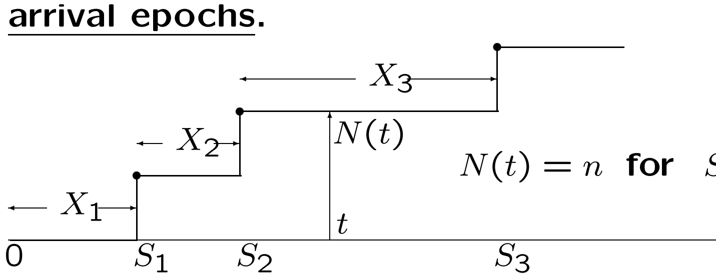
Def: A sequence $Z_1, Z_2, \dots,$ of rv's converges in mean square to a rv Z if $\lim_{n\to\infty} \mathsf{E}\left[|Z_n - Z|^2\right] = 0$.

Convergence in mean square implies convergence in probability implies convergence in distribution.

<u>Arrival processes</u>

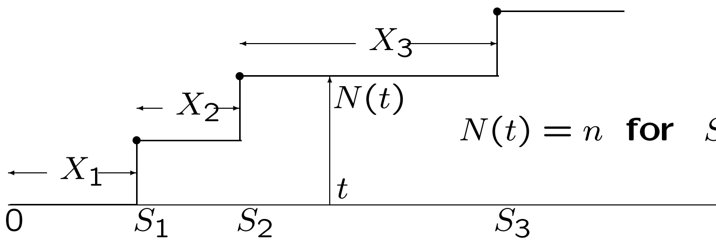
Def: An arrival process is a sequence of increasing rv's $0 < S_1 < S_2 < \cdots$ where $S_{i-1} < S_i$ means that $S_i - S_{i-1} = X_i$ is a positive rv, i.e., $F_{X_i}(0) = 0$.

The differences $X_i = S_i - S_{i-1}$ for $i \ge 2$ and $X_1 = S_1$ are called <u>interarrival times</u> and the S_i are called <u>arrival epochs</u>.



$N(t) = n$ for $S_n \le t < S_{n+1}$

For each $t > 0$, $N(t)$ is the number of arrivals in $(0, t]$. We call $\{N(t);\ t > 0\}$ an <u>arrival counting process</u>.

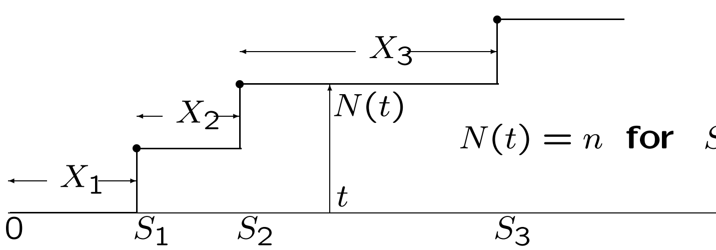


$N(t) = n$ **for** $S_n \le t < S_{n+1}$

A sample path or sample function of the process is a sequence of sample values, $S_1 = s_1, S_2 = s_2, \ldots$ **.**

Each sample path corresponds to a particular stair case function and the process can be viewed as the ensemble (with joint probability distributions) of such stair case functions.

The figure shows how the arrival epochs, interarrival times, and counting variables are interrelated for a generic stair case function.



$N(t) = n$ **for** $S_n \le t < S_{n+1}$

It can also be seen that any sample path can be specified by the sample values of $N(t)$ **for all** t**, by** S_i **for all** i**, or by** X_i **for all** i**, so that essentially an arrival process can be specified by the counting process, the interarrival times, or the arrival epochs.**

The major relation we need to relate the counting process $\{N(t); t > 0\}$ **to the arrival process is**

$$\{S_n \le t\} = \{N(t) \ge n\}; \qquad \textbf{for all } n \ge 1, t > 0.$$

If $S_n = \tau$ **for some** $\tau \le t$**, then** $N(\tau) = n$ **and** $N(t) \ge n$**.**

Although stochastic processes are usually defined by a sequence of rv's or a family of rv's indexed by the reals, we represent arrival processes by arrival epochs, interarrival intervals, or counting variables, which ever is convenient at the moment.

The general class of arrival processes is too complicated to make much further progress. We simplify as follows:

Def: A renewal process is an arrival process for which the interarrival intervals $X_1, X_2, \ldots$ are IID.

Def: A Poisson process is a renewal process for which each X_i has an exponential distribution, $\mathsf{F}_X(x) = 1 - \exp(-\lambda x)$ for $x \geq 0$, where λ is a fixed parameter called the rate.

Poisson processes and exponential rv's.

The remarkable simplicity of Poisson processes is closely related to the 'memoryless' property of the exponential rv.

Def: A rv X is memoryless if X is positive and, for all real $t > 0$ and $x > 0$,

$$\Pr\{X > t + x\} = \Pr\{X > t\} \Pr\{X > x\}. \qquad (1)$$

Since the interarrival interval for a Poisson process is exponential, i.e., $\Pr\{X > x\} = \exp(-\lambda x)$ for $x > 0$.

$$\exp(-\lambda(t + x)) = \exp(-\lambda t) \exp(-\lambda x).$$

Thus X is memoryless.

Thm: A rv X is memoryless if and only if it is exponential. (see text)

The reason for the word 'memoryless' is more apparent when using conditional probabilities,

$$\Pr\{X > t + x \mid X > t\} = \Pr\{X > x\}$$

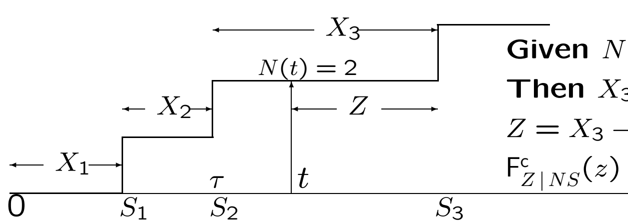
If people in a checkout line have exponential service, and you have waited 15 minute for the person in front, what is his or her remaining service time?

Exponential, same as when service started. The remaining service time 'doesn't remember' the elapsed time. Has your time waiting been wasted?

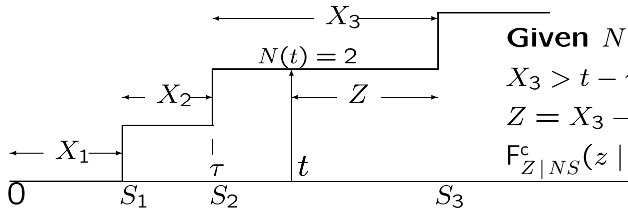
Why do you move to another line if someone takes a long time?

Thm: For a Poisson process of rate λ, and any given $t > 0$, the interval Z from t to the next arrival after t has distribution $\mathsf{F}^{\mathsf{c}}_Z(z) = \exp(-\lambda z)$ for all $z > 0$. The rv Z is independent of $N(t)$, and, given $N(t) = n$, Z is independent of $S_1, \dots, S_n$ and of $\{N(\tau); 0 < \tau < t\}$. (Thm 2.2.1 in text, but not stated very well there).

Idea of proof: Condition on $N(t) = n$ and $S_n = \tau$, i.e., the number n of arrivals in $(0, t]$ and the time, τ of the most recent arrival in $(0, t]$.



Given $N(t)=2, S_2 = \tau$.
Then $X_3 > t - \tau$ and
$Z = X_3 - (t - \tau)$
$\mathsf{F}^{\mathsf{c}}_{Z\mid NS}(z) = \exp(-\lambda z)$



Given $N(t)=2, S_2 = \tau$,
$X_3 > t - \tau$ and
$Z = X_3 - (t - \tau)$
$\mathsf{F}^{c}_{Z\,|\,NS}(z \mid n, \tau)) = \exp(-\lambda z)$

The conditional distribution of Z does not vary with the conditioning values, $N(t) = n$ and $S_n = \tau$, so Z is stat. independent of $N(t)$ and $S_{N(t)}$ (see text).

The rv $S_{N(t)}$ is the time of the last arrival up to and including t. A given sample point ω maps into $N(t)(\omega) = n$, say, and then into $S_n = \tau$. We find the distribution function of $S_{N(t)}$ much later, but don't need it here.

This theorem essentially extends the idea of memorylessness to the entire Poisson process. That is, starting at any $t > 0$, the interval Z to the next arrival is an exp rv of rate λ. Z is independent of everything before t.

Subsequent interarrival times are independent of Z and of the past. Thus the interarrival process starting at t with first interarrival Z, and continuing with subsequent interarrivals is a Poisson process.

The counting process corresponding to this interarrival process is $N(t') - N(t)$ for $t' > t$. This is a Poisson process shifted to start at time t, i.e., for each t', $N(t') - N(t)$ has the same distribution as $N(t' - t)$. Same for joint distributions.

This new process is independent of $\{N(\tau); 0 < \tau \leq t.\}$

Stationary and independent increments

Def: A counting process $\{N(t); t > 0\}$ has the stationary increment property if $N(t') - N(t)$ has the same distribution as $N(t' - t)$ for all $t' > t > 0$.

Stationary increments means that the distribution of the number of arrivals in the increment $(t, t']$ is a function only of $t' - t$. The distribution depends on the length of the interval, but not the starting time.

Let $\widetilde{N}(t, t') = N(t') - N(t)$, i.e., $\widetilde{N}(t, t')$ is the number of arrivals in the increment $(t, t']$. Thus stationary increments means that $\widetilde{N}(t, t')$ has the same distribution as $N(t' - t)$.

Poisson processes have the stationary increment property.

Def: A counting process $\{N(t); t > 0\}$ has the independent increment property if, for every $t_1, t_2, \ldots, t_n$, the rv's $N(t_1), \widetilde{N}(t_1, t_2), \ldots \widetilde{N}(t_{n-1}, t_n)$ are independent.

This implies that the number of arrivals in each of a set of non-overlapping intervals are independent rv's.

For a Poisson process, we have seen that $\widetilde{N}(t_{i-1}, t_i)$ is independent of $\{N(\tau);\, \tau \le t_{i-1}\}$, so Poisson processes have independent increments.

Thm: Poisson processes have stationary and independent increments.

<u>The Erlang and Poisson distributions</u>

For a Poisson process of rate λ, the PDF of arrival epoch S_2 can be found by convolving the density of X_1 and X_2. Thus

$$\begin{aligned} \mathsf{f}_{S_2}(t) &= \int_0^t [\lambda \exp(-\lambda x)][\lambda \exp(-\lambda(t-x))]dx \\ &= \lambda^2 t \exp(-\lambda t) \end{aligned}$$

Using iteration and convolving $\mathsf{f}_{S_{n-1}}(t)$ with $\lambda \exp(-\lambda t)$ for each n,

$$\mathsf{f}_{S_n}(t) = \frac{\lambda^n t^{n-1} \exp(-\lambda t)}{(n-1)!}$$

This is called the Erlang density.

Stopping to smell the roses while doing all this computation will be very helpful.

The joint density of $X_1, \dots, X_n$ is

$$\begin{aligned} \mathsf{f}_{\vec{X}^n}(x_1, \dots, x_n) &= \lambda^n \exp(-\lambda x_1 - \lambda x_2 - \cdots - \lambda x_n) \\ &= \lambda^n \exp(-\lambda s_n) \quad \text{where } s_n = \sum_{i=1}^n x_i \end{aligned}$$

$$\mathsf{f}_{\vec{S}^n}(s_1, \dots, s_n) = \lambda^n \exp(-\lambda s_n)$$

This says that the joint density is uniform over the region where $s_1 < s_2 < \cdots < s_n$. Given that the nth arrival is at s_n, the other $n-1$ arrivals are uniformly distributed in $(0, s_n)$, subject to the ordering.

Integrating (or looking ahead), we get the Erlang marginal density.

Thm: For a Poisson process of rate λ, the PMF for $N(t)$ is the Poisson PMF,

$$\mathsf{p}_{N(t)}(n) = \frac{(\lambda t)^n \exp(-\lambda t)}{n!}$$

Pf: We will calculate $\Pr\Big\{t < S_{n+1} \le t+\delta\Big\}$ in two ways and go to the limit $\delta \to 0$. First we use the density for S_{n+1} to get

$$\Pr\Big\{t < S_{n+1} \le t+\delta\Big\} = \mathsf{f}_{S_{n+1}}(t)(\delta + \mathbf{o}(\delta))$$

where $\lim_{\delta\to 0} \frac{\mathbf{o}(\delta)}{\delta} = 0$. Next we use the independent increment property over $(0,t]$ and $(t, t+\delta]$.

$$\Pr\Big\{t < S_{n+1} \le t+\delta\Big\} = \mathsf{p}_{N(t)}(n)(\lambda\delta + \mathbf{o}(\delta)) + \mathbf{o}(\delta)$$

Equating and going to the limit, $\mathsf{p}_{N(t)}(n) = \mathsf{f}_{S_{n+1}}(t)/\lambda$.

$$\mathsf{p}_{N(t)}(n) = \frac{(\lambda t)^n \exp(-\lambda t)}{n!}; \qquad \textbf{Poisson PMF}$$

Note that the Poisson PMF is a function of λt and not of λ or t separately. This is the probability of n arrivals in an interval of size t with rate λ.

If we measure length in a different system of units, λ will change accordingly, so the Poisson PMF has to be a function of λt only.

Note also that $N(t) = N(t_1) + \widetilde{N}(t_1, t)$ for any $0 < t_1 < t$. Thus $N(t)$ is the sum of 2 independent rv's, one with the Poisson distribution for t_1 and the other for $t - t_1$.

This extends to any k disjoint intervals, which is one reason the Poisson counting process is so easy to work with.

Alternate definitions of Poisson process

Is it true that any arrival process for which $N(t)$ has the Poisson PMF for a given λ and for all t is a Poisson process of rate λ?

As usual, the marginal PMF's alone are not enough. The joint distributions must also be those of the Poisson process (see text).

Thm: If an arrival process has the stationary and independent increment properties and if $N(t)$ has the Poisson PMF for given λ and all $t > 0$, then the process is Poisson.

VHW Pf: The stationary and independent increment properties show that the joint distribution of arrivals over any given set of disjoint intervals is that of a Poisson process. Clearly this is enough.

Is it true that any arrival process with the stationary and independent increment properties is a Poisson process?

These properties capture much of our intuition about Poisson processes, but they allow bulk processes to sneak through, i.e., processes in which simultaneous arrivals are possible. Poisson processes satisfy the following condition for very small increments:

$$\Pr\left\{\widetilde{N}(t, t+\delta) = n\right\} = \begin{cases} 1 - \lambda\delta + o(\delta) & \text{for} \quad n = 0 \\ \lambda\delta & \text{for} \quad n = 1 \\ o(\delta) & \text{for} \quad n \geq 2 \end{cases}$$

Thm: If an arrival process has the stationary and independent increment properties and satisfies the above incremental condition, then it is a Poisson process.

Relation to Bernoulli process

Bernoulli processes are often viewed as the discrete time version of Poisson processes. There is a confusing feature here that must be cleared up first

The binary rv's $Y_1, Y_2, \ldots$ of the Bernoulli process have no direct analogs in the Poisson process.

When we view a Bernoulli process as an arrival process, an arrival occurs at discrete time n if and only if $Y_n = 1$. Thus $S_n = Y_1 + \cdots + Y_n$ is the number of arrivals up to and including time n. Thus $\{S_n; n \geq 1\}$ is analogous to the Poisson counting process $\{N(t); t > 0\}$.

The interarrival intervals, $X_1, X_2, \ldots$ for a Bernoulli arrival process are the intervals between successive 1's in the binary stream. Thus $X_1 = k$ if $Y_i = 0$ for $1 \leq i \leq k-1$ and $Y_k = 1$. Thus $\mathsf{p}_{X_1}(k) = p(1-p)^{k-1}$ for all $k \geq 1$. Subsequent interarrival intervals are IID with X_1.

Thus, the interarrival intervals for the Bernoulli counting process are geometrically distributed.

The Bernoulli counting process is defined only at integer times, but if we consider the arrivals within integer intervals, we see that the stationary and independent increment properties are satisfied (over those integer values).

We can clearly extend the definition of a Bernoulli counting process to a shrunken Bernoulli counting process where changes occur at intervals of δ rather than unit intervals.

Consider a sequence of shrinking Bernoulli processes, holding λ/δ constant but shrinking λ and δ. The geometric interarrival interval becomes exponential and the Bernoulli counting process converges to the Poisson counting process. (see text).