## Problem Set 6

### Multirate Systems, Polyphase Structures, Linear Prediction and All-Pole Modeling

**Issued:** Tuesday October 18, 2005.                    **Due:** Tuesday October 25, 2005.

**Reading:** Multirate systems and polyphase decomposition: Section 4.7. Linear prediction and all-pole modeling: notes handed out in class.

Note that Project II relies heavily on polyphase implementations of the various filters. A good time to start Project II (or start thinking about it) is while you are completing Project IB and are learning about polyphase structures.
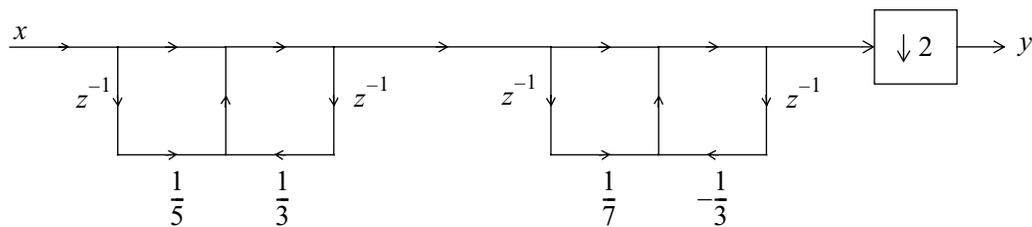
### Problem 6.1

OSB Problem 4.52

### Problem 6.2

OSB Problem 4.53

### Problem 6.3

Consider the following system:



(a) How many multiplications per output sample are needed on average (not counting factors of $\pm 1$)?

(b) Use the noble identity to push the downsampler toward the input of the system as much as possible.

(c) How many multiplications per output sample are needed on average for your new system in part (b) (not counting factors of $\pm 1$)?

## Problem 6.4

For each system shown below, design an efficient implementation. Take "efficient" to mean having a small number of multiplications per input sample and a small number of delay elements. Make block diagram manipulations in a step-by-step manner with brief justifications of your steps. Make sure to to fully specify any filters, *i.e.*, write out the transfer functions.

(a) $x_a[n] \rightarrow \boxed{H_a(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4}} \rightarrow \boxed{\downarrow 3} \rightarrow y_a[n]$

(b) $x_b[n] \rightarrow \boxed{H_b(z) = b_0 + b_1 z^{-6} + b_2 z^{-12}} \rightarrow \boxed{\downarrow 3} \rightarrow \boxed{\uparrow 3} \rightarrow \boxed{\downarrow 2} \rightarrow y_b[n]$

(c) $x_c[n] \rightarrow \boxed{H_c(z) = c_0 + c_1 z^{-2} + c_2 z^{-6} + c_3 z^{-12}} \rightarrow \boxed{\downarrow 6} \rightarrow \boxed{\uparrow 3} \rightarrow y_c[n]$

## Problem 6.5

In the system below, $w[n]$ is a real, zero-mean, white, wide-sense stationary random sequence with variance $\sigma_w^2$, and $h[n]$ is LTI.

$$w[n] \rightarrow \boxed{\uparrow 2} \xrightarrow{\ x[n]\ } \boxed{h[n]} \xrightarrow{\ y[n]\ } \boxed{\downarrow 2} \rightarrow d[n]$$

(a) Simplify this system as much as possible. As a first step, perform a polyphase decomposition on $h[n]$.

(b) Find an expression for the autocorrelation of $d[n]$ *i.e.*, $R_{dd}[n, m] = \mathbf{E}\,[d[n]d[n + m]]$, in terms of $\sigma_w^2$ and $h[n]$ **for all values** of $n$. Use your result from part (a) as a first step.

## Problem 6.6

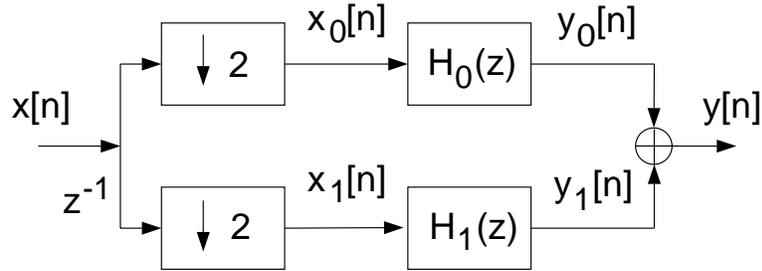Consider the decimation filter structure shown below:
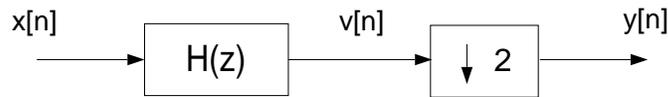


Figure 6.6-1:

where $y_0[n]$ and $y_1[n]$ are generated according to the following forward recursions:

$$y_0[n] = \frac{1}{4}y_0[n-1] - \frac{1}{3}x_0[n] + \frac{1}{8}x_0[n-1]$$

$$y_1[n] = \frac{1}{4}y_1[n-1] + \frac{1}{12}x_1[n]$$

(a) How many multiplies per output sample does the implementation of the filter structure require? Consider a divide to be equivalent to a multiply.

The decimation filter can also be implemented as shown below,



where $v[n] = av[n-1] + bx[n] + cx[n-1]$.

(b) Determine $a$, $b$, and $c$.

(c) How many multiplies per output sample does this second implementation require?

## Problem 6.7

Consider the signal

$$s[n] = 2 \left(\frac{1}{3}\right)^n u[n] + 3 \left(-\frac{1}{2}\right)^n u[n].$$

We wish to model this signal using a second-order $(p = 2)$ all-pole model, or equivalently using second-order linear prediction.

For this problem, since we are given an analytical expression for $s[n]$ and $s[n]$ is the impulse response of an all-pole filter, we can obtain the linear prediction coefficients directly from the Z-transform of $s[n]$ (you're asked to do this in part (a)). In practical situations, we are typically given data, i.e. a set of signal values, and not an analytical expression. In this case, even when the signal to be modeled is the impulse response of an all-pole filter, we need to perform some computation on the data, such as the methods discussed in the lecture and in the notes, in order to determine the linear prediction coefficients.

There are also situations in which an analytical expression is available for the signal but the signal is not the impulse response of an all-pole filter and we would like to model it as such, in which case, we again need to carry out computations such as those discussed in lecture and the notes.

(a) Determine the linear prediction coefficients $a_1, a_2$ directly from the Z-transform of $s[n]$.

(b) Write the normal equations for $p = 2$ to obtain equations for $a_1, a_2$ in terms of $\phi_s[m]$.

(c) Find the values of $\phi_s[0]$, $\phi_s[1]$, and $\phi_s[2]$ for the signal $s[n]$ given above.

(d) Solve the system of equations from part (a) using the values you found in part (b) to get values for the $a_k$'s. You may find MATLAB or another computer tool useful to solve these equations.

(e) Are the values of $a_k$ from part (c) what you would expect for this signal? Justify your answer clearly.

(f) Suppose you wish to model the signal now with $p = 3$. Write the normal equations for this case.

(g) Find the value of $\phi_s[3]$.

(h) Solve for the values of $a_k$ when $p = 3$. You may find MATLAB or another computer tool useful to solve these equations.

(i) Are the values of $a_k$ found in part (h) what you would expect given $s[n]$? Justify your answer clearly.

(j) Would the values of $a_1, a_2$ you found in (h) change if we model the signal with $p = 4$?