# System Identification

## 6.435

<u>SET 3</u>

– Nonparametric Identification

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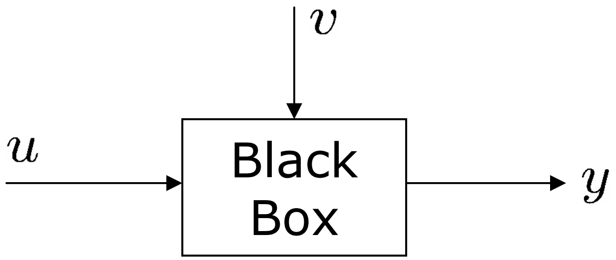
# Nonparametric Methods for System ID

- Time domain methods
  - Impulse response
  - Step response
  - Correlation analysis / time
- Frequency domain methods
  - Sine-wave testing
  - Correlation analysis / Frequency
  - Fourier-analysis
  - Spectral analysis

# Problem Formulation




- Actual system $G_o$ is Linear time-invariant stable.
- Process:

$$\begin{aligned} y(t) &= G_o u(t) + v(t) \\ &= g_o * u(t) + v(t) \end{aligned}$$

- Time domain methods $\Rightarrow$ estimates of $g_o$
- Frequency-domain methods $\Rightarrow$ estimates of $G_o\left(e^{i\omega}\right)$.

- Tests:

a) $\left|G_o\left(e^{i\omega}\right) - \widehat{G}\left(e^{i\omega}\right)\right|$ at each freq.

b) $|g_o(t) - \widehat{g}(t)| \quad \forall \quad t \geq 0$

c) $\sum_{t=0}^{\infty} |g_o(t) - \widehat{g}(t)|$

d) $\sup_{\omega} \left|G_o\left(e^{i\omega}\right) - \widehat{G}\left(e^{i\omega}\right)\right|$

# Time-Domain Methods

- Impulse response $u = \alpha\delta(t)$

$$\Rightarrow \quad y = \alpha g_o(t) + v(t)$$

estimate: $\hat{g}(t) = \dfrac{y(t)}{\alpha}$

Analysis: $|g_o(t) - \hat{g}(t)| = \dfrac{|v(t)|}{\alpha}$ small if $\alpha >> 1$.

Practicality: not very useful.

- Step response $u = \alpha \quad \forall \quad t \geq 0$

$$\Rightarrow \quad y(t) = \alpha \sum_{k=0}^{\infty} g_o(t) + v(t)$$

| estimate: $\widehat{g}(t) = \dfrac{y(t) - y(t-1)}{\alpha}$ |
|---|

Analysis: $$|g_o(t) - \widehat{g}(t)| = \frac{|v(t) - v(t-1)|}{\alpha}$$

Practicality: Not good for determining $g_o(t)$. Good for determining delays, modes….

# Methods (Continued)

- Correlation Analysis

$$y(t) = g_o * u + v$$

- Assume $u$ is quasi-stationary

  $u, v$ are uncorrelated.

- $\bar{E}y(t)u(t-\tau) = R_{yu}(\tau) = g_o * R_u(\tau) = \sum_{k=1}^{\infty} g_o(k) R_u(k-\tau)$

- Case I: If $u \sim WN \quad \Rightarrow \quad R_{yu} = \alpha g_o * \delta(z) = \alpha g_o.$

  To estimate:

$$R_{yu}^N(\tau) = \frac{1}{N} \sum_{t=\tau}^{N} y(t)u(t-\tau)$$

$$R_u^N(\tau) = \frac{1}{N} \sum_{t=\tau}^{N} u(t)u(t-\tau)$$

$$\alpha = \quad R_u^N(0) = \quad \frac{1}{N} \sum_{t=0}^{N} u^2(t)$$

$$\Rightarrow \quad \hat{g}(\tau) = \frac{\frac{1}{N} \sum_{t=\tau}^{N} y(t)u(t-\tau)}{\frac{1}{N} \sum_{t=0}^{N} u^2(t)}$$

- Case II: Input is not white.

$$R_{yu}(\tau) = g_o * R_u(\tau)$$

Using the approximation

$$R_{yu}^N(\tau) = \hat{g} * R_u^N(\tau)$$

In matrix form:

$$\begin{pmatrix} R_{yu}^N(0) \\ \vdots \\ R_{yu}^N(M-1) \end{pmatrix} = \begin{pmatrix} R_u^N(0) & R_u^N(-1) & R_u^N(-(M-1)) \\ R_u^N(1) & R_u^N(0) & R_u^N(-(M-2)) \\ \vdots & & \vdots \\ R_u^N(M-1) & \dots & R_u^N(0) \end{pmatrix} \begin{pmatrix} \hat{g}(0) \\ \vdots \\ \hat{g}(M-1) \end{pmatrix}$$

notice $R_u^N(\tau) = R_u^N(-\tau)$.

$\Rightarrow$ Estimate $\widehat{g}(\tau) = \sum_{k=0}^{M-1} \widehat{g}(k) q^{-k}.$

- Question: Under what conditions the above system has a unique solution? Persistency of excitation!

- Note that you get the same estimate regardless of the spectrum of the noise.

# Analysis of Correlation Method

- Estimate

$$\widehat{h}(\tau) = \frac{\frac{1}{N}\sum_{t=1}^{N} y(t)u(t-\tau)}{\frac{1}{N}\sum_{t=1}^{N} u^2(t)}$$

- $E\left(\widehat{h}(\tau)\right) \to h(\tau)$ as $N \to \infty$

- Need to determine the covariance of $\widehat{h}(\tau) - h(\tau)$ for a fixed large N.

- $\hat{h}(k)-h(k) \simeq \frac{1}{R_u(0)}\frac{1}{N}\left[\sum_{t=1}^{N}\{y(t+k)-h(k)u(t)\}u(t)\right]$

$$= \frac{1}{\sigma^2 N}\sum_{t=1}^{N}\left(\sum_{\substack{i=0\\ i\neq k}}^{\infty}(h(i)u(t+k-i)+v(t+k))u(t)\right)$$

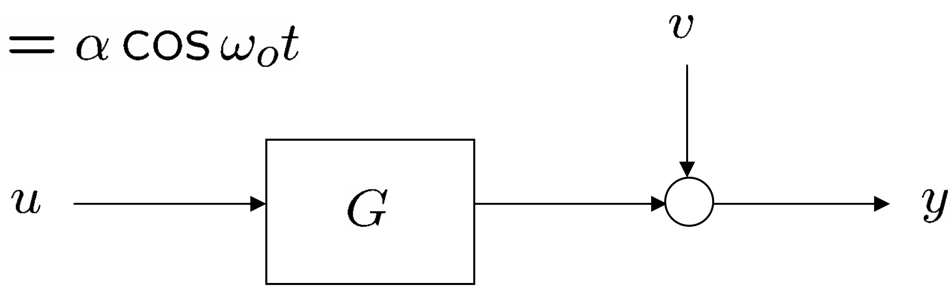
- $E\left(\hat{h}(\nu)-h(\nu)\right)\left(\hat{h}(\mu)-h(\mu)\right) \simeq \frac{R_v(\mu-v)}{N\sigma^2}+\frac{1}{N}\sum_{i=0}^{\infty}h(i)h(i+|v-\mu|)$

$$+\frac{1}{N}\sum_{\tau=-\mu}^{\nu}h(\tau+\mu)h(\nu-\tau)-\frac{2}{N}h(\mu)h(\nu)$$

- Covariance, proportional to $\frac{1}{N}$.

# Frequency-Response Analysis

- Input $u(t) = \alpha \cos \omega_o t$




$$y(t) = \alpha \left| G\left(e^{i\omega_o}\right) \right| \cos(\omega_o t + \phi) + v(t) + \text{transients}$$

$$\phi = \angle G\left(e^{i\omega_o}\right).$$

- Extract $\left| G\left(e^{i\omega_o}\right) \right|, \phi \Rightarrow \widehat{G}_N\left(e^{i\omega_o}\right)$

- How do you measure $\left| G\left(e^{i\omega_o}\right) \right|, \Phi$ in the presence of noise?
  A good approach is correlation.

- Define

$$I_C(N) = \frac{1}{N}\sum_{t=1}^{N} y(t)\cos\omega_o t \qquad I_S(N) = \frac{1}{N}\sum_{t=1}^{N} y(t)\sin\omega_o t$$

- $$I_C(N) = \frac{1}{2N}\sum_{t=1}^{N} \alpha\left|G\left(e^{i\omega_o}\right)\right| [\cos\phi + \cos(2\omega_o t + \phi)]$$

$$+ \quad \frac{1}{N}\sum_{t=1}^{N} v(t)\cos\omega_o t \quad + \quad \text{transients}$$

$$\longrightarrow \quad \frac{2}{\alpha}\left|G\left(e^{i\omega_o}\right)\right|\cos\phi$$

- $$I_S(N) \quad \longrightarrow \quad -\frac{2}{\alpha}\left|G\left(e^{i\omega_o}\right)\right|\sin\phi$$

- Estimate:

$$\left|G\left(e^{i\omega_o}\right)\right| = \frac{2}{\alpha}\sqrt{I_c^2 + I_s^2}$$

$$\phi = -\tan^{-1}\frac{I_S(N)}{I_C(N)}$$

- Comment: $$Y_N(\omega) = \frac{1}{\sqrt{N}}\sum_{t=1}^{N} y(t)e^{-i\omega t}$$

$$Y_N(\omega) = (I_C - iI_S)\sqrt{N}$$

$$U_N(\omega) = \frac{\sqrt{N}\alpha}{2}$$

$$\Rightarrow \quad \widehat{G}_N\left(e^{i\omega}\right) = \frac{Y_N(\omega)}{U_N(\omega)}$$

# Empirical Transfer Function Estimate (ETFE)

- For an arbitrary input

$$\hat{\hat{G}}\left(e^{i\omega}\right) = \frac{Y_N(\omega)}{U_N(\omega)} \quad \text{when} \quad U_N(\omega) \neq 0$$

- Recall: Correlation analysis

$$\hat{G}\left(e^{i\omega}\right) = \frac{\Phi_{yu}(\omega)}{\Phi_u(\omega)}$$

- If $u = e^{i\frac{2\pi}{N}k}$, then the previous analysis shows that

$$\hat{G}\left(e^{i\omega}\right) = \hat{\hat{G}}\left(e^{i\omega}\right) \quad \omega = \frac{2\pi}{N}k$$

- Similarly for $u$ = White input.

- General Procedure

1. Calculate $\hat{\hat{G}}\left(e^{i\frac{2\pi}{N}k}\right) \quad , \quad k = 1, \ldots, N$

2. Obtain the inverse DFT:

$$\hat{\hat{g}}(t) = \frac{1}{N} \sum_{k=1}^{N} \hat{\hat{G}}\left(e^{i\frac{2\pi}{N}k}\right) e^{i\frac{2\pi}{N}tk} \quad , \quad t = 1, \ldots, N$$

3. Define $\hat{\hat{G}}(q) = \sum_{t=1}^{N} \hat{\hat{g}}(t) q^{-t}$

- The algorithm is quite efficient; requires only the computation of the Inverse DFT. Note also that the algorithm is Linear.

# Properties of EFTE

Theorem:

Given: $y = Gu + v$

With:

- $|u(t)| \leq C$
- $s(t)$ is stationary, zero mean with spectrum $\Phi_v$
- $\hat{\hat{G}}_N\left(e^{i\omega}\right) = \dfrac{Y_N(\omega)}{U_N(\omega)}$

Then:

1. $E\left(\widehat{\widehat{G}}_N\left(e^{i\omega}\right)\right) = G_o\left(e^{i\omega}\right) + \frac{\rho_1(N)}{U_N(\omega)}$

2. $E\left(\widehat{\widehat{G}}_N\left(e^{i\omega}\right) - G_o\left(e^{i\omega}\right)\right)\left(\widehat{\widehat{G}}_N\left(e^{-i\xi}\right) - G_o\left(e^{-i\xi}\right)\right)$

$$= \begin{cases} \frac{1}{|U_N(\omega)|^2}\left[\Phi_v(\omega) + \rho_2(N)\right] & \xi = \omega \\ \frac{\rho_2(N)}{U_N(\omega)U_N(-\xi)} & \xi - \omega = \pm\frac{2\pi}{N}k, \quad 1 \le k \le N-1 \end{cases}$$

$$|\rho_2(N)| \le \frac{C_2}{\sqrt{N}}$$

# Proofs

- Bias

$$\hat{\hat{G}}\left(e^{i\omega}\right) = \frac{Y_N(\omega)}{U_N(\omega)} = G\left(e^{i\omega}\right) + \frac{R_N(\omega)}{U_N(\omega)} + \frac{V_N(\omega)}{U_N(\omega)}$$

$$E\left(\hat{\hat{G}}\left(e^{i\omega}\right)\right) = G\left(e^{i\omega}\right) + \frac{R_N(\omega)}{U_N(\omega)}$$

- Covariance

1st: Compute $E\left(V_N(\omega)V_N(-\xi)\right)$

$$E\left(V_N(\omega)V_N(-\xi)\right) = E\frac{1}{N}\sum_{r=1}^{N}\sum_{s=1}^{N} v(r)e^{-i\omega r}v(s)e^{+i\xi s}$$

$$= \frac{1}{N}\sum_{r=1}^{N}\sum_{s=1}^{N} R_v(r-s)e^{+i(\xi s-\omega r)}$$

$\tau = r - s$

$$= \frac{1}{N}\sum_{r=1}^{N}\sum_{\tau=r-1}^{N-1} R_v(\tau)e^{i\xi r-i\xi\tau}e^{-i\omega r}$$

$$= \frac{1}{N}\sum_{r=1}^{N} e^{i(\xi-\omega)r}\sum_{\tau=r-1}^{r-N} R_v(\tau)e^{-i\xi\tau}$$

- $$\sum_{\tau=r-1}^{r-N} R_v(\tau)e^{-i\xi\tau} = \Phi_v(\xi) - \sum_{\tau=-\infty}^{\tau-N-1} R_v(\tau)e^{-i\xi\tau} - \sum_{\tau=r}^{\infty} R_v(\tau)e^{-i\xi\tau}$$
- $$\frac{1}{N}\sum_{r=1}^{N} e^{i(\xi-\omega)r} = \begin{cases} 1 & \text{if } \xi = \omega \\ 0 & \text{if } \xi - \omega = \pm\frac{2\pi}{N}k, \quad k = 1, \ldots, N-1 \end{cases}$$
- $$\left|\frac{1}{N}\sum_{r=1}^{N} e^{i(\xi-\omega)r} \sum_{\tau=-\infty}^{\tau-N-1} R_v(\tau)e^{-i\xi\tau}\right| \leq \frac{1}{N}\sum_{r=1}^{N}\sum_{\tau=-\infty}^{\tau-N-1} |R_v(\tau)|\left|e^{-i\xi\tau}\right|$$

$$= \frac{1}{N}\sum_{\tau=-\infty}^{-1}\sum_{r=\tau+N+1}^{N} |R_v(\tau)|$$

$$\leq \frac{1}{N}\sum_{\tau=-\infty}^{-1} \tau\,|R_v(\tau)|$$

$$\leq \frac{C}{N}$$

$$C = \sum_{\tau=-\infty}^{\infty} \tau\,|R_v(\tau)|$$

- Put together

$$E\left(V_N(\omega)V_N(-\xi)\right) = \begin{cases} \Phi_v(\omega) + \rho_2(N) & \omega = \xi \\ \rho_2(N) & \omega - \xi = \pm\frac{2\pi}{N}k, \quad 1 \le k \le N-1 \end{cases}$$

$$\rho_2(N) \le \frac{2C}{N}$$

Now:

$$E\left(\widehat{\widehat{G}}\left(e^{i\omega}\right) - G\left(e^{i\omega}\right)\right)\left(\widehat{\widehat{G}}\left(e^{-i\xi}\right) - G\left(e^{-i\xi}\right)\right)$$

$$= E\left(\frac{V_N(\omega)}{U_N(\omega)}\frac{V_N(-\xi)}{U_N(-\xi)}\right) - E\left(\frac{R_N(\omega)}{U_N(\omega)}\frac{R_N(-\xi)}{U_N(-\xi)}\right)$$

$$= \begin{cases} \frac{1}{|U_N(\omega)|^2}\left[\Phi_v(\omega) + \rho_2(N)\right] & \xi = \omega \\ \frac{\rho_2(N)}{U_N(\omega)U_N(-\xi)} & \xi - \omega = \frac{2\pi}{N}k, \quad 1 \le k \le N-1 \end{cases}$$

# Comments on EFTE

- Suppose $U$ = periodic

  $|U_N(\omega)|^2$ increases as a function of $N$ for some $\omega = \frac{2\pi}{N}k$ and zero for others

  — EFTE is defined for a fixed number of frequencies, i.e. independent of $N$.

  — At these frequencies, ETFE is unbiased and Covariance decays as $\frac{1}{N}$ . (Recall $R_N = 0$ ).

- Suppose $V$ is a stochastic process, uncorrelated with $v$

$$|U_N(\omega)|^2 \xrightarrow{\text{in dist.}} \Phi_u(\omega) \quad \text{(a bounded function)}$$

  - ETFE is asymptotically unbiased, with increasingly more well-defined frequencies (as $N \to \infty$).
  - The variance does not decrease as $N \to \infty$.
  - Estimates are asymptotically uncorrelated.

# Spectral Estimation

- Traditionally

$$\{v(1), \quad \ldots \quad , v(N)\} \rightsquigarrow \Phi_v$$

estimate

N-Long time series

- In here, different context.

$$\hat{\hat{G}}_N\left(e^{i\omega}\right) \rightsquigarrow \hat{G}_N\left(e^{i\omega}\right)$$

{smaller variance}

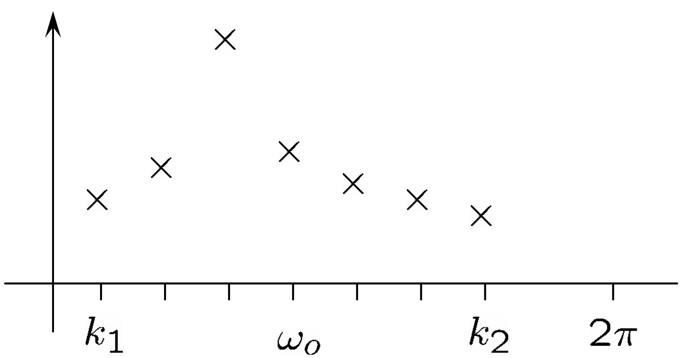
- Theme:
  - Show the mechanics
  - Importance of windowing, tradeoffs
  - Relate to spectral estimation

# Spectral Estimation: Non Std (Ljung)

- Idea: the actual function $G\left(e^{i\omega}\right)$ is smooth. The values of $G\left(e^{i\omega}\right)$ should be related for small intervals $\omega$.

- According to previous analysis, $\hat{\hat{G}}\left(e^{i\omega}\right)$ is uncorrelated with $\hat{\hat{G}}\left(e^{-i\xi}\right)$ and has variance

$$\frac{\Phi_v(\omega)}{|U_N(\omega)|^2}$$




- Suppose $\omega_o$ satisfies

$$\frac{2\pi}{N}k_1 = \omega_o - \Delta\omega < \omega_o < \omega_o + \Delta\omega = \frac{2\pi}{N}k_2$$

- Define the estimate (new) at $\omega_o$ as follows:

$$\hat{G}_N\left(e^{i\omega_o}\right) = \frac{\sum_{k=k_1}^{k_2} \alpha_k \hat{\hat{G}}_k\left(e^{i\frac{2\pi}{N}k}\right)}{\sum_{k=k_1}^{k_2} \alpha_k}$$

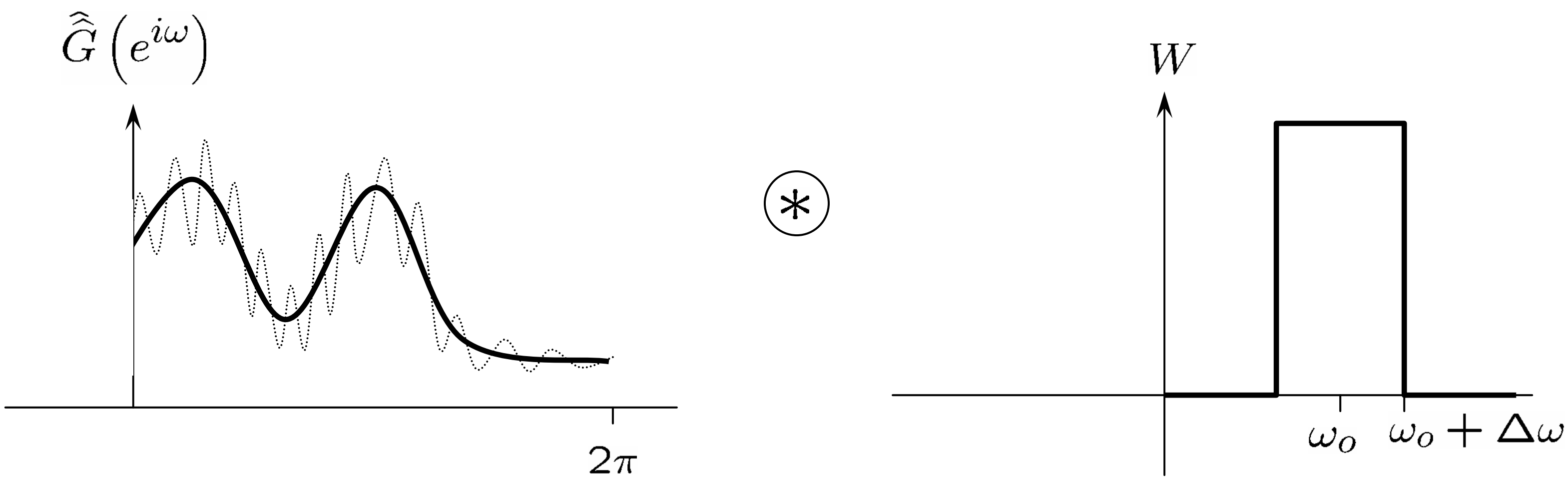
- Where $\alpha_{k1}, \ldots, \alpha_{kn}$ are chosen so that $E\left(\hat{G}_N\left(e^{i\omega_o}\right) - G\left(e^{i\omega_o}\right)\right)^2$ is minimized.

- Solution:

$$\alpha_k = \frac{\left|U_N\left(\frac{2\pi}{N}k\right)\right|^2}{\Phi_v\left(\frac{2\pi}{N}k\right)}$$

- As $N \to \infty$, the sums $\rightsquigarrow$ integrals

$$\hat{G}_N\left(e^{i\omega}\right) = \frac{\displaystyle\int_{\omega_o-\Delta\omega}^{\omega_o+\Delta\omega} \alpha(\xi)\hat{\hat{G}}_k\left(e^{i\xi}\right)d\xi}{\displaystyle\int_{\omega_o-\Delta\omega}^{\omega_o+\Delta\omega} \alpha(\xi)d\xi}$$



- Equivalently: Let $W_\gamma(\xi)$ be a window function. Then,

$$\widehat{G}_N\left(e^{i\omega}\right) = \frac{\int_{-\pi}^{\pi} W_\gamma(\omega-\xi)\alpha(\xi)\widehat{\widehat{G}}\left(e^{i\xi}\right)d\xi}{\int_{-\pi}^{\pi} W_\gamma(\omega-\xi)\alpha(\xi)d\xi}$$

- If $\Phi_v$ is unknown, but slowly varying in frequency

$$\widehat{G}_N = \frac{\int_{-\pi}^{\pi} W_\gamma(\omega-\xi)\left|U_N(\xi)\right|^2\widehat{\widehat{G}}\left(e^{i\xi}\right)d\xi}{\int_{-\pi}^{\pi} W_\gamma(\omega-\xi)\left|U_N(\xi)\right|^2 d\xi}$$

# Relations to Traditional Spectral Analysis

- Recall:

$$|U_N(\xi)|^2 \xrightarrow[\text{in distribution}]{} \Phi_u(\xi)$$

$$\Rightarrow \int_{-\pi}^{\pi} W_\gamma(\omega-\xi)\,|U_N(\xi)|^2\,d\xi \longrightarrow \int_{-\pi}^{\pi} W_\gamma(\omega-\xi)\Phi_u(\xi)d\xi$$

estimate of $\Phi_u$

$$\simeq \Phi_u(\quad)$$

- Define:

$$\Phi_u^N(\omega) \triangleq \int_{-\pi}^{\pi} W_\gamma(\omega - \xi) \left|U_N(\xi)\right|^2 d\xi$$

- $\left|U_N(\xi)\right|^2 \hat{\hat{G}}\left(e^{i\xi}\right) = U_N^*(\xi) Y_N(\xi)$

Similarly:

$$\Phi_{yu}^N(\omega) = \int_{-\pi}^{\pi} W_\gamma(\omega - \xi) U_N^*(\xi) Y_N(\xi) d\xi$$

- Conclusion

$$\hat{G}\left(e^{i\omega}\right) = \frac{\Phi_{yu}^N(\omega)}{\Phi_u^N(\omega)}$$

# Efficient Computation

- $R_u^N(\tau) = \frac{1}{N}\sum_{t=1}^{N} u(t)u(t-\tau)$

  $W_\gamma(\omega) \longleftrightarrow W_\gamma(\tau)$

  $$\Rightarrow \Phi_u^N(\omega) = \sum_{\tau=-\infty}^{\infty} W_\gamma(\tau) R_u^N(\tau) e^{-i\omega\tau}$$

  Of course $W_\gamma(\tau) \simeq 0$ for $\tau$ large enough but not as large as $N$. Example is:

  $$W_\gamma(\tau) = 1 - \frac{|\tau|}{\gamma} \qquad 0 \leq \tau \leq \gamma. \qquad \text{(Bartlett)}$$

- Similarly for $R_{yu}^N$

# Analysis of Spectral Estimation

- $$\hat{G}_N\left(e^{i\omega}\right) = \frac{\int_{-\pi}^{\pi} W_\gamma(\xi-\omega_o)\,|U_N(\xi)|^2\,\hat{\hat{G}}_N\left(e^{i\xi}\right)d\xi}{\int_{-\pi}^{\pi} W_\gamma(\xi-\omega_o)\,|U_N(\xi)|^2\,d\xi}$$

  and $\hat{\hat{G}}_N\left(e^{i\xi}\right) = G\left(e^{i\xi}\right) + \frac{R_N(\xi)}{U_N(\xi)} + \frac{V_N(\xi)}{U_N(\xi)}$

- $$E\left(\hat{G}_N\left(e^{i\omega_o}\right)\right) \simeq \frac{\int_{-\pi}^{\pi} W_\gamma(\xi-\omega_o)\,|U_N(\xi)|^2\,G\left(e^{i\xi}\right)d\xi}{\int_{-\pi}^{\pi} W_\gamma(\xi-\omega_o)\,|U_N(\xi)|^2\,d\xi}$$

  $$\simeq \frac{\int_{-\pi}^{\pi} W_\gamma(\xi-\omega_o)\Phi_u(\xi)G\left(e^{i\xi}\right)d\xi}{\int_{-\pi}^{\pi} W_\gamma(\xi-\omega_o)\Phi_u(\xi)d\xi}$$

- Write

$$\Phi_u(\xi) = \Phi_u(\omega_o) + (\xi - \omega_o)\Phi_u^{'}(\omega_o) + \frac{1}{2}(\xi - \omega_o)^2 \Phi_u^{''}(\omega_o)$$

$$G\left(e^{i\xi}\right) = G\left(e^{i\omega_o}\right) + (\xi - \omega_o) G^{'}\left(e^{i\omega_o}\right) + \frac{1}{2}(\xi - \omega_o)^2 G^{''}\left(e^{i\omega_o}\right)$$

- Recall: $\int_{-\pi}^{\pi} W_\gamma(\xi) d\xi = 1$ $\quad$ $\int_{-\pi}^{\pi} \xi W_\gamma(\xi) d\xi = 0$

$$\int_{-\pi}^{\pi} \xi^2 W_\gamma(\xi) = M(\gamma) \to 0 \quad \text{as} \quad \gamma \to \infty$$

$$\int_{-\pi}^{\pi} W_\gamma^2(\xi) d\xi = \bar{W}(\gamma) \to \infty \quad \text{as} \quad \gamma \to \infty$$

- $E\left(\widehat{G}_N\left(e^{i\omega_o}\right)\right) \simeq \dfrac{\text{Numerator}}{\text{Denominator}}$

Numerator: $\Phi_u(\omega_o)G\left(e^{i\omega_o}\right)+M(\gamma)\left[\Phi_u^{'}G_o^{'}+\dfrac{G_o^{''}\Phi_u}{2}+\dfrac{\Phi_u^{''}G_o}{2}\right]$

Denominator: $\Phi_u(\omega_o)+\dfrac{M(\gamma)}{2}\Phi_u^{''}(\omega_o)$

- $E\left(\widehat{G}_N\left(e^{i\omega_o}\right)\right) \cong G\left(e^{i\omega_o}\right)+M(\gamma)\left[\dfrac{1}{2}G_o^{''}\left(e^{i\omega_o}\right)+G_o^{'}\left(e^{i\omega_o}\right)\dfrac{\Phi_u^{'}}{\Phi_u}(\omega_o)\right]$

$\Rightarrow$ for each finite $\gamma$, the estimate is biased.

- $\widehat{G}_N - E\widehat{G}_N = \dfrac{\int_{-\pi}^{\pi} W_\gamma(\xi-\omega_o)\,|U_N(\xi)|^2 \left[\dfrac{V_N(\xi)}{U_N(\xi)}\right]}{\int_{-\pi}^{\pi} W_\gamma(\xi-\omega_o)\,|U_N(\xi)|^2\, d\xi}$

- $E\left|\widehat{G}_N - E\widehat{G}_N\right|^2 = \dfrac{\frac{2\pi}{N}\int_{-\pi}^{\pi} W_\gamma^2(\xi-\omega_o)\Phi_u(\xi)\Phi_v(\xi)d\xi}{\Phi_u\left(e^{i\omega_o}\right) + \frac{M(\gamma)}{2}\Phi_u''(\omega_o)}$

$$\cong \quad \frac{1}{N}\cdot\frac{\bar{W}(\gamma)\Phi_u\Phi_v(\omega_o)}{(\Phi_u(\omega_o))^2}$$

- For a fixed $\gamma$, $\quad \text{Var}\left(\widehat{G}_N\right) \to 0 \quad \text{as} \quad N \to \infty.$

- Improved variance on the expense of the biase.

# Estimating the Disturbance Spectrum

- $y(t) = G_o u(t) + v(t)$

- If $v(t)$ was measurable, then

$$\hat{\Phi}_v^N(\omega_o) = \int_{-\pi}^{\pi} W_\gamma(\xi - \omega_o)\,|V_N(\xi)|^2)d\xi$$

<u>Bias</u>: $$E\Phi_v^N \cong \Phi_v(\omega_o) + \frac{M(\gamma)}{2}\Phi_v''(\omega_o)$$

Variance $$E\left[\Phi_v^N - E\Phi_v^N\right]^2 \simeq \frac{W(\gamma)}{N}\Phi_v^2(\omega)$$

- <u>Problem</u>: $v(t)$ is not readily measurable.

- The residual spectrum.

$\widehat{G}_N(q)$ is the estimate $\quad\quad \widehat{v}(t) = y(t) - \widehat{G}_N(q)u$

- $\widehat{\Phi}_v^N(\omega_o) = \int_{-\pi}^{\pi} W_\gamma(\xi - \omega_o) \left| Y_N(\xi) - \widehat{G}_N\left(e^{i\xi}\right) U_N(\xi) \right|^2 d\xi$

$$\simeq \widehat{\Phi}_y(\omega_o) - \frac{\left|\Phi_{yu}^N(\omega)\right|^2}{\Phi_u^N(\omega)}$$

- Define

$$\widehat{k}_{yu}^N(\omega) = \sqrt{\frac{\left|\widehat{\Phi}_{yu}^N\right|^2}{\widehat{\Phi}_y^N(\omega)\widehat{\Phi}_u^N(\omega)}}$$

- $\widehat{\Phi}_v^N(\omega) = \widehat{\Phi}_y^N(\omega)\left(1 - \left(\widehat{k}_{yu}^N(\omega)\right)^2\right)$